Coordinator
Yilin Wang

Probability Theory

Exercise Sheet 4

Exercise 4.1

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables in a probability space $(\Omega, \mathcal{A}, P)$. Define the two sequences of random variables $(Y_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ by

$$Y_n := \min_{1\leq i\leq n} X_i \quad \text{and} \quad M_n := \max_{1\leq i\leq n} X_i$$

(a) Let X_1 be uniformly distributed on the interval $[0,1]$. Show that nY_n converges in distribution to an exponential random variable Z with parameter 1, i.e., the density of Z is $e^{-x}1_{[0,\infty)}(x), x \in \mathbb{R}$.

(b) Let X_1 be exponentially distributed with parameter 1. Show that $M_n - \log n$ converges in distribution to a random variable Z with Gumbel distribution, i.e. the density of Z is $e^{-x}\exp(-e^{-x}), x \in \mathbb{R}$.

Exercise 4.2

(a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables converging in probability to a random variable X. Show that $(X_n)_{n\in\mathbb{N}}$ converges to X in distribution.

(b) The converse does not hold in general. Instead, show that if the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a *constant* random variable $X = c$, then $(X_n)_{n\in\mathbb{N}}$ converges in probability to c.

Exercise 4.3

(a) Let f be a (not necessarily Borel-measurable) function from $\mathbb{R}$ to $\mathbb{R}$. Show that the set of discontinuities of f, defined as

$$U_f := \left\{x \in \mathbb{R} \mid f \text{ is discontinuous in } x\right\},$$

is Borel-measurable.

(b) Assume that $X_n \to X$ in distribution. Let f be measurable and bounded, such that $P[X \in U_f] = 0$. Use (2.2.13) – (2.2.14) from the lecture notes to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)].$$

(c) Let f be measurable and bounded on $[0, 1]$, with U_f of Lebesgue measure 0. Show that the corresponding Riemann sums converge to the integral of f, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \underset{n\to\infty}{\to} \int_0^1 f(x)\mathrm{d}x.$$

Submission deadline: 13:15, Oct. 24.

Location: During exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Yilin Wang
He-Lang	Tue 13-14	ML H 41.1	Angelo Abächerli
Lanz-Sa	Tue 14-15	HG F 26.5	Vincenzo Ignazio
Sch-Zh	Tue 14-15	ML H 41.1	Lukas Gonon