ETH Zurich HS 2017
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 6

Exercise 6.1 Let X and Y be two independent Bernoulli distributed random variables with parameter p. Define $Z = 1_{\{X+Y=0\}}$ and $\mathcal{G} = \sigma(Z)$. Find $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$. Are these random variables also independent?

Exercise 6.2 Let X and Y be random variables whose joint distribution is the uniform distribution on the triangle $\{(x, y) \in \mathbb{R}^2 : \ 0 \leq y \leq x \leq 1\}$.

(a) Compute the distribution of Y/X.

(b) Show that Y/X and X are independent.

(c) Compute the conditional expectation $E[Y|X]$.

Exercise 6.3

(a) Let $Z_n, n \geq 1$, and $Y_n, n \geq 1$, be random variables defined on the same probability space, and assume $Z_n \xrightarrow{d} Z$ and $Y_n \xrightarrow{P} 0$. Show that $Z_n + Y_n \xrightarrow{d} Z$

Hint: Recall the proof of problem 4.2(a).

(b) (*Random Index Central Limit Theorem*) Let X_i, $i \in \mathbb{N}$, be i.i.d. random variables with $E[X_i] = 0$, $E[X_i^2] = \sigma^2 \in (0, \infty)$. Furthermore, let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $a_n \in \mathbb{N}$ for all n and $a_n \to \infty$, and $(N_n)_{n\in\mathbb{N}}$ a sequence of $\mathbb{N}$–valued random variables, such that $N_n/a_n \to 1$ in probability. Let $S_n := \sum_{i=1}^{n} X_i$ for $n \in \mathbb{N}$. Show that
$$\frac{S_{N_n}}{\sigma\sqrt{a_n}} \text{ converges to } \mathcal{N}(0, 1) \text{ in distribution.}$$

Hint: Show that $\frac{S_{a_n}}{\sigma\sqrt{a_n}}$ converges in distribution to $\mathcal{N}(0, 1)$ and, using Kolmogorov's inequality, that
$$\frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}} \xrightarrow[n\to\infty]{P} 0.$$

Submission deadline: 13:15, Nov 7.

Location: During exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Yilin Wang
He-Lang	Tue 13-14	ML H 41.1	Angelo Abächerli
Lanz-Sa	Tue 14-15	HG F 26.5	Vincenzo Ignazio
Sch-Zh	Tue 14-15	ML H 41.1	Lukas Gonon