ETH Zurich HS 2017
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 7

Exercise 7.1 *Polya's Urn:* An urn initially contains s black and w white balls. We consider the following process. At each step a random ball is drawn from the urn, and is replaced by t balls of the same colour, for some fixed $t \geq 1$. We define the random variable Y_n as the proportion of black balls in the urn after the n-th iteration. Show that $E[Y_{n+1}|\sigma(Y_1, Y_2, \dots Y_n)] = Y_n$, for all $n \in \mathbb{N}$, that is, $\{Y_n\}_{n\in\mathbb{N}}$ is a martingale.

Exercise 7.2 Let X be a random variable in $L^2(\Omega, \mathcal{A}, P)$ and $\mathcal{F} \subseteq \mathcal{A}$. The *conditional variance* of X given $\mathcal{F}$ is defined as $\mathrm{Var}[X|\mathcal{F}] := E[(X - E[X|\mathcal{F}])^2|\mathcal{F}]$. Prove that

(a) $\mathrm{Var}[X|\mathcal{F}] = E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2$;

(b) $\mathrm{Var}(X) = E[\mathrm{Var}[X|\mathcal{F}]] + \mathrm{Var}[E[X|\mathcal{F}]]$.

(c) Compute $\mathrm{Var}[X|\mathcal{F}]$, where $\mathcal{F} = \sigma(A_1, A_2)$ where $\{A_1, A_2\}$ is a partition of Ω and $P(A_i) > 0$ for $i = 1, 2$.

Exercise 7.3

Let S be a random variable with $P[S > t] = e^{-t}$, for all $t > 0$. Calculate the following conditional expectations for arbitrary $t > 0$:

(a) $E\Big[S \;\Big|S \wedge t\Big]$, where $S \wedge t := \min(S, t)$;

(b) $E\Big[S \;\Big|S \vee t\Big]$, where $S \vee t := \max(S, t)$.

Submission deadline: 13:15, Nov 14.

Location: During exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Yilin Wang
He-Lang	Tue 13-14	ML H 41.1	Angelo Abächerli
Lanz-Sa	Tue 14-15	HG F 26.5	Vincenzo Ignazio
Sch-Zh	Tue 14-15	ML H 41.1	Lukas Gonon