

Probability Theory

Exercise Sheet 9

Exercise 9.1 Let (Ω, \mathcal{F}, P) be a probability space with a filtration $(\mathcal{F}_n)_{n \geq 0}$. Let $S \leq T$ be two bounded $(\mathcal{F}_n)_{n \geq 0}$ -stopping times and let $(X_n)_{n \geq 0}$ be an $(\mathcal{F}_n)_{n \geq 0}$ -submartingale. Show that

$$E[X_T | \mathcal{F}_S] \geq X_S, P\text{-a.s.}$$

Exercise 9.2 An urn initially contains one red and one green ball. At each time $n = 1, 2, 3, \dots$ a random ball is drawn, and then is put back together with an additional ball of the same colour. Let Z_n be the number of red balls in the urn before the n -th ball is drawn (so that $Z_1 = 1$).

- (a) Show that $\left(\frac{Z_n}{n+1}\right)_{n \in \mathbb{N}}$ is a martingale for the natural filtration.
- (b) Show that $\left(\frac{Z_n}{n+1}\right)_{n \in \mathbb{N}}$ converges P -almost surely and in L^1 to a random variable X . Determine the distribution of X .

Hint: Recall that $\frac{1}{n} \sum_{k=1}^n \delta_{\frac{k}{n+1}} \xrightarrow{d} \lambda$, as $n \rightarrow \infty$, where λ is the Lebesgue measure on $[0, 1]$ and δ_x is the Dirac measure of x .

Exercise 9.3 Consider a probability space (Ω, \mathcal{F}, P) equipped with a filtration $\{\mathcal{F}_n\}_{n \geq 0}$, and let X_n be an \mathcal{F}_n -martingale for which $|X_{n+1} - X_n| \leq M$ P -a.s. for some fixed $M < \infty$. Define the events C, D by

$$C := \{\lim X_n \text{ exists and is finite}\},$$
$$D := \{\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty\}.$$

Show that $P[C \cup D] = 1$.

Hint: Show that $P[C^c \cap (\{\sup_{n \in \mathbb{N}} X_n < a\} \cup \{\inf_{n \in \mathbb{N}} X_n > -a\})] = 0$, for all $a > 0$, by considering the processes $\{X_{T_A \wedge n}\}_{n \geq 0}$, for $A = [a, \infty)$ and $A = (-\infty, -a]$, where $T_A = \inf\{n \geq 0 : X_n \in A\}$.

Submission deadline: 13:15, Nov 28.

Location: During exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Yilin Wang
He-Lang	Tue 13-14	ML H 41.1	Angelo Abächerli
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