

# Probability Theory

## Exercise Sheet 10

**Exercise 10.1** Consider a Galton-Watson process (see p. 97 of the lecture notes)  $Z_n, n \geq 0$ , with offspring distribution  $\nu = \text{Bin}(2, p), p \in [0, 1]$ . We are interested in the probability  $\vartheta(p) = P[Z_n > 0, \forall n \geq 0]$  that the population does not go extinct. Show that

$$\vartheta(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq 1/2; \\ \frac{2p-1}{p^2} & \text{if } 1/2 < p \leq 1. \end{cases}$$

*Hint:* One way to prove this is to use the results for the various cases (subcritical, critical, supercritical) from Section 3.5 A), pp. 97-101 of the lecture notes.

**Exercise 10.2** *The generalized Borel-Cantelli lemma:* Consider  $(\Omega, \mathcal{F}, P)$  with filtration  $\{\mathcal{F}_n\}_{n \geq 0}$ , and let  $A_n \in \mathcal{F}_n, n \geq 1$ , be a sequence of events. Show that, up to a  $P$ -nullset,

$$\limsup_{n \rightarrow \infty} A_n = \left\{ \sum_{n \geq 1} P[A_n | \mathcal{F}_{n-1}] = \infty \right\}.$$

*Hint:* Use Exercise 9.3.

**Exercise 10.3** *Probabilistic solution to the discrete Dirichlet problem:* Let  $A \subseteq \mathbb{Z}^d$  be finite,  $f : \mathbb{Z}^d \setminus A \rightarrow \mathbb{R}$  any function, and  $(S_n)_{n \in \mathbb{N}_0}$  a simple random walk on  $\mathbb{Z}^d$  with starting point  $S_0 = 0$ . For  $x \in \mathbb{Z}^d$  let  $T_x := \inf\{n \in \mathbb{N}_0 \mid x + S_n \notin A\}$ . Finally, let  $\mathcal{F}_n := \sigma(S_0, \dots, S_n)$  and  $g(x) := E[f(x + S_{T_x})]$ .

- (a) Show that  $T_x < \infty$   $P$ -a.s. Thus  $f(x + S_{T_x})$  exists a.s. *Hint:* Use Exercise 9.3.
- (b) Show that  $g$  solves the discrete Dirichlet problem on  $A$  with boundary condition  $f$ , i.e.,

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{Z}^d \setminus A \\ \frac{1}{2d} \sum_{\substack{\|y-x\|=1 \\ y \in \mathbb{Z}^d}} g(y) & \text{if } x \in A. \end{cases}$$

- (c) Show that  $E[f(x + S_{T_x}) | \mathcal{F}_1] = g(x + S_{T_x \wedge 1})$   $P$ -a.s.

**Submission deadline:** 13:15, Dec 05.

**Location:** During exercise class or in the tray outside of HG E 65.

**Office hours (Präsenz):** Mon. and Thu., 12:00-13:00 in HG G 32.6.

**Class assignment:**

| Students | Time & Date | Room      | Assistant        |
|----------|-------------|-----------|------------------|
| An-Gr    | Tue 13-14   | HG F 26.5 | Yilin Wang       |
| He-Lang  | Tue 13-14   | ML H 41.1 | Angelo Abächerli |
| Lanz-Sa  | Tue 14-15   | HG F 26.5 | Vincenzo Ignazio |
| Sch-Zh   | Tue 14-15   | ML H 41.1 | Lukas Gonon      |