ETH Zurich HS 2017
D-MATH, D-PHYS
Prof. A.-S. Sznitman

Coordinator
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Probability Theory

Exercise Sheet 11

Exercise 11.1 Let $(X_n)_{n\geq 0}$ be a uniformly integrable family of random variables on $(\Omega, \mathcal{A}, P)$.

(a) Assume that X_n converges to a random variable X in distribution. Show that

$$E[X_n] \xrightarrow{n\to\infty} E[X].$$

Remark: Compare to (3.6.18)–(3.6.20), p. 112 of the lecture notes.

(b) Assume that X_n converges to a random variable X in probability. Show that $X \in L^1$ and that X_n converges to X in L^1.

Exercise 11.2 Let X_n, $n \geq 0$, be a uniformly integrable submartingale and N a stopping time.

(a) Show that $\sup_n E[X^+_{N\wedge n}] \leq \sup_n E[X^+_n] < \infty$.

(b) Show that X_N (where $X_N 1_{\{N=\infty\}} = 1_{\{N=\infty\}} \lim_n X_n$) is integrable.

(c) Show that $X_{N\wedge n}$, $n \geq 0$, is a uniformly integrable submartingale.

Exercise 11.3 Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent, non-negative random variables with expectation 1. Consider the natural filtration $(\mathcal{F}_n)_{n\geq 0}$. We define

$$M_0 = 1, \quad M_n = Y_1 Y_2 \cdots Y_n, \text{ for } n \in \mathbb{N}.$$

(a) Prove that $(M_n)_{n\in\mathbb{N}_0}$ is a non-negative martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ and there exists a random variable M_∞, so that $M_n \to M_\infty$ a.s.

Let $a_n := E[\sqrt{Y_n}]$.

(b) Show that $a_n \in (0, 1]$.

(c) Show that if $\prod_n a_n > 0$, it holds that $M_n \to M_\infty$ in L^1 and $E[M_\infty] = 1$.

(d) Show that if $\prod_n a_n = 0$, then $M_\infty = 0$ a.s.

Submission deadline: 13:15, Dec 12.

Location: During exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Yilin Wang
He-Lang	Tue 13-14	ML H 41.1	Angelo Abächerli
Lanz-Sa	Tue 14-15	HG F 26.5	Vincenzo Ignazio
Sch-Zh	Tue 14-15	ML H 41.1	Lukas Gonon