ETH Zurich HS 2017
D-MATH, D-PHYS
Prof. A.-S. Sznitman

Coordinator
Yilin Wang

Probability Theory

Exercise Sheet 12

Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical (time-homogenous) Markov chain with a *countable* state space E, a transition kernel K, and canonical coordinates $(X_n)_{n\geq 0}$. The matrix

$$Q = (Q(x,y))_{x,y\in E} := (K(x,\{y\}))_{x,y\in E} = (P_x[X_1 = y])_{x,y\in E}$$

is then called the *transition matrix* of the Markov chain.

Exercise 12.1 Let E be a countable set, $(S, \mathcal{S})$ a measurable space, $(Y_n)_{n\geq 1}$ a sequence of i.i.d. S-valued random variables. We define a sequence $(X_n)_{n\geq 0}$ through $X_0 = x \in E$ and $X_{n+1} = \Phi(X_n, Y_{n+1})$, where $\Phi : E \times S \to E$ is a measurable map. Show that $(X_n)_{n\geq 0}$ induces a time-homogenous Markov chain and calculate the corresponding transition matrix.

Exercise 12.2 Let $(X_n)_{n\geq 0}$ be a sequence of random variables with values in $[0,1]$. We set $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$. Suppose that $X_0 = a \in [0,1]$ and

$$P\left[X_{n+1} = \frac{X_n}{2}\Big|\mathcal{F}_n\right] = 1 - X_n, \qquad P\left[X_{n+1} = \frac{1+X_n}{2}\Big|\mathcal{F}_n\right] = X_n.$$

(a) Show that $(X_n)_{n\geq 0}$ is a $\mathcal{F}_n$-martingale that converge to a random variable X_∞ P-almost surely and in L^2.

(b) Show that $E\left[(X_{n+1} - X_n)^2\right] = \frac{1}{4}E\left[X_n(1 - X_n)\right]$.

Exercise 12.3 Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration, and let $(M_n)_{n\geq 0}$ be a non-negative $(\mathcal{F}_n)$-martingale such that $M_{n+1} \leq CM_n$ for all n. Let $a > 0$ and set $T := \inf\{n \geq 0 |\ M_n > a\}$.
Show that

(a) There exists some random variable M_∞, such that $M_n \to M_\infty$ P-a.s.

(b) $E\left[M_n 1_{\{0<T\leq n\}}\right] \leq CaP[0 < T \leq n]$.

(c) $E\left[M_n 1_{\{\sup_{0\leq k\leq n} M_k > a\}}\right] \leq E\left[M_0 1_{\{M_0>a\}}\right] + CaP\left[\sup_{0\leq k\leq n} M_k > a\right]$.

(d) $E[M_n \log^+ M_n] \leq E[M_0 \log^+ M_0] + CE\left[\sup_{0\leq k\leq n} M_k\right]$, where $\log^+ x := (\log x) \vee 0$.
Hint: Modify the expression from **c)** and integrate.

(e) If $E\left[\sup_{n\in\mathbb{N}} M_n\right] < +\infty$ and $M_0 \log^+ M_0 \in L^1$, then

i) $\sup_{n\geq 0} E[M_n \log^+ M_n] < +\infty$.

ii) $M_\infty \log^+ M_\infty \in L^1$.

iii) $M_n \to M_\infty$ not only P-a.s. but also in L^1.

Submission deadline: 13:15, Dec 19.

Location: During exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Yilin Wang
He-Lang	Tue 13-14	ML H 41.1	Angelo Abächerli
Lanz-Sa	Tue 14-15	HG F 26.5	Vincenzo Ignazio
Sch-Zh	Tue 14-15	ML H 41.1	Lukas Gonon