ETH Zürich, Fall 2017
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Mathematical Foundations for Finance

Exercise sheet 13

Please hand in your solutions until Tuesday, 19/12/2017, 18:00 into your assistant's box next to HG G 53.2.

Exercise 13.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $M = (M_t)_{t\geq 0}$ be a local $(P, \mathbb{F})$-martingale and $W = (W_t)_{t\geq 0}$ a $(P, \mathbb{F})$-Brownian motion.

(a) Let $H = (H_t)_{t\geq 0}$ be in $L^2(M)$. Compute $E\left[\int_0^T H_s dM_s\right]$ and $\mathrm{Var}\left[\int_0^T H_s dM_s\right]$. How do the expressions look for $M := W$?

(b) Let $H_s := \exp(-4s)$. Show that $\int_0^T H_s dW_s$ is in fact normally distributed. What are the mean and the variance of this normal distribution? How would the result change if $H : \mathbb{R} \to \mathbb{R}$ were an arbitrary (deterministic) continuous function?
Hint 1: Use the dominated convergence theorem for stochastic integrals from page 93 in the lecture notes.
Hint 2: If $X_n \sim \mathcal{N}\left(\mu_n, \sigma_n^2\right)$, $X_n \to X$ in probability, $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2 > 0$, then $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

(c) By coming up with a counterexample, show that the normality of $\int_0^T H_s dW_s$ from (b) does not hold for an arbitrary continuous $H \in L^2(W)$.

Exercise 13.2 Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ are given by

$$d\widetilde{S}^0_t = \widetilde{S}^0_t r dt \quad \text{and} \quad d\widetilde{S}^1_t = \widetilde{S}^1_t \left(\mu dt + \sigma dW_t\right), \tag{1}$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$ are deterministic.

(a) Prove using Itô's formula that the discounted stock price process $S^1 = \widetilde{S}^1/\widetilde{S}^0$ solves

$$dS^1_t = S^1_t \left((\mu - r)dt + \sigma dW_t\right). \tag{2}$$

(b) Prove using Itô's formula that

$$S^1 = \left(S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right)\right)_{t\in[0,T]},$$

i.e. show that the process $\left(S^1_0 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right)\right)_{t\in[0,T]}$ solves (2).

(c) Let $L^\lambda := -\lambda W$ and $Z^\lambda := \mathcal{E}(L^\lambda)$. Prove that the process $W^\lambda := (W_t + \lambda t)_{t\in[0,T]}$ is a Brownian motion under the measure Q_λ given by $\frac{dQ_\lambda}{dP} := Z^\lambda_T$.

(d) Prove that for the right choice of λ, the discounted stock price process S^1 is a Q_λ-martingale.
Hint: Rewrite $\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t$ as function of $W^\lambda_t, t, \sigma, \mu,$ and r.

Exercise 13.3 Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \le s \le T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ are given by

$$\frac{d\widetilde{S}^0_t}{\widetilde{S}^0_t} = r dt \quad \text{and} \quad \frac{d\widetilde{S}^1_t}{\widetilde{S}^1_t} = \mu dt + \sigma dW_t,$$

with $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$ deterministic. Using the notation of the previous exercise, denote $Q^* := Q_{\lambda^*}$, where λ^* is the unique value of λ making Q_λ an equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$.
Hint: If you did not find λ^* in Exercise 13.2 (d), you can use that $\lambda^* = \frac{\mu - r}{\sigma}$.

(a) Hedge the *square option*, i.e., find a self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ such that

$$V_0 + \int_0^T \vartheta_u dS^1_u = \frac{(\widetilde{S}^1_T)^2}{\widetilde{S}^0_T}.$$

Hint: Look for a representation result under Q^*, not under P.

(b) Hedge the *inverted option*, i.e., find a self-financing strategy $\varphi \mathrel{\widehat{=}} (\overline{V}_0, \overline{\vartheta})$ such that

$$\overline{V}_0 + \int_0^T \overline{\vartheta}_u dS^1_u = \frac{1}{\widetilde{S}^0_T \widetilde{S}^1_T}.$$