

Mathematical Foundations for Finance

Exercise sheet 1

This sheet contains material which is **fundamental** for this course and assumed to be known. Please hand in your solutions until Tuesday, 26/09/2017, 18:00 into your assistant's box next to HG G 53.2.

Exercise 1.1 Let (Ω, \mathcal{F}, P) be a probability space with $\Omega := \{UU, UD, DD, DU\}$, $\mathcal{F} := 2^\Omega$ and P defined by $P[\omega] := 1/4$ for all $\omega \in \Omega$. Let $Y_1, Y_2 : \Omega \rightarrow \mathbb{R}$ be two random variables with $Y_1(UU) = Y_1(UD) := 2$, $Y_1(DD) = Y_1(DU) := 1/2$, $Y_2(UU) = Y_2(DU) := 2$ and $Y_2(DD) = Y_2(UD) := 1/2$. Define the process $X = (X_k)_{k=0,1,2}$ by

$$X_k(\omega) = X_0 \prod_{i=1}^k Y_i(\omega) \quad \text{for } k = 0, 1, 2,$$

with $X_0 = 8$ P -a.s.

- Explicitly write down the sequences of σ -fields $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ and $\mathbb{G} = (\mathcal{G}_k)_{k=0,1,2}$ defined by $\mathcal{F}_k := \sigma(X_i, 1 \leq i \leq k)$ and $\mathcal{G}_k := \sigma(X_k)$, $k = 0, 1, 2$.
- Show that $Z : \Omega \rightarrow \mathbb{R}$ defined by $Z(\omega) := 2X_1(\omega) + 1$ is $\sigma(X_1)$ -measurable.
- Do \mathbb{F} and \mathbb{G} form filtrations on (Ω, \mathcal{F}) ? Why or why not?
- Is X adapted to \mathbb{F} or \mathbb{G} (in case any of the former is a filtration on (Ω, \mathcal{F}))?
- Try to give financial interpretations for X and \mathbb{F} .

Exercise 1.2 Let (Ω, \mathcal{F}, P) be a probability space and $X : \Omega \rightarrow \mathbb{R}$ a random variable with $X \geq 0$ P -a.s. Prove that $E[X] = 0$ implies that $X = 0$ P -a.s.

Hint: Try using one of the basic limit theorems from probability/measure theory.

Exercise 1.3 Let (Ω, \mathcal{F}, P) be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ -field. Then the P -a.s. unique random variable Z such that

- Z is \mathcal{G} -measurable and integrable,
- $E[X\mathbf{1}_A] = E[Z\mathbf{1}_A]$ for all $A \in \mathcal{G}$,

is called *conditional expectation of X given \mathcal{G}* and is denoted by $E[X|\mathcal{G}]$. (This is the formal definition of conditional expectation of X given \mathcal{G} ; see Section 8.2 in the lecture notes.)

- Use the definition above to show that if X is \mathcal{G} -measurable, then $E[X|\mathcal{G}] = X$ P -a.s.
- Use the definition of conditional expectation to show that $E[E[X|\mathcal{G}]] = E[X]$.
- Use the definition of conditional expectation to show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$, i.e. if \mathcal{G} is P -trivial, then $E[X|\mathcal{G}] = E[X]$ P -a.s.
- Consider an integrable random variable Y on (Ω, \mathcal{F}, P) , and two constants $a, b \in \mathbb{R}$. Use the definition of conditional expectation to show that $E[aX + bY|\mathcal{G}] = aE[X|\mathcal{G}] + bE[Y|\mathcal{G}]$ P -a.s.

- (e) Suppose that \mathcal{G} is generated by a finite partition of Ω , i.e. there exists a collection $(A_i)_{i=1}^n$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \dots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \dots, n$. Use the definition of conditional expectation to show that

$$E[X | \mathcal{G}] = \sum_{i=1}^n E[X | A_i] \mathbb{1}_{A_i} \quad P\text{-a.s.}$$

Hint 1: Recall that $E[X | A_i] = E[X \mathbb{1}_{A_i}] / P[A_i]$ and try to write X as a sum of random variables each of which only takes non-zero values on a single A_i .

Hint 2: Check that any set $A \in \mathcal{G}$ is of the form $\bigcup_{j \in J} A_j$ for some $J \subseteq \{1, \dots, n\}$.