ETH Zürich, Fall 2017
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Mathematical Foundations for Finance

Exercise sheet 2

Please hand in your solutions until Tuesday, 03/10/2017, 18:00 into your assistant's box next to HG G 53.2.

Exercise 2.1 Let us assume the basic multiplicative model for our financial market $(\widetilde{S}^0, \widetilde{S}^1)$. We start on a probability space $(\Omega, \mathcal{F}, P)$ with random variables $r_1, \dots, r_T > -1$ and $Y_1, \dots, Y_T > 0$ for a $T \in \mathbb{N}$. Define for $k = 0, \dots, T$

$$\widetilde{S}^0_k := \prod_{j=1}^{k} (1 + r_j), \quad \widetilde{S}^1_k := S^1_0 \prod_{j=1}^{k} Y_j,$$

with a constant $S^1_0 > 0$.

(a) A natural filtration to use in this model is the filtration generated by $Y = (Y_k)_{k=1,\dots,T}$ and $r = (r_k)_{k=1,\dots,T}$, i.e. the one given by $\mathcal{F}'_k = \sigma(Y_1, \dots, Y_k, r_1, \dots, r_k)$ for $k = 0, \dots, T$. Show that if one assumes r to be predictable with respect to this filtration, then we have that $\mathcal{F}'_k = \mathcal{F}_k := \sigma(\widetilde{S}^1_0, \widetilde{S}^1_1, \dots, \widetilde{S}^1_k)$.
(*Hint: if $\mathcal{A}$ and $\mathcal{B}$ are two collections of subsets of Ω, then $\sigma(\mathcal{A} \cup \mathcal{B}) = \sigma(\sigma(\mathcal{A}) \cup \sigma(\mathcal{B}))$*)

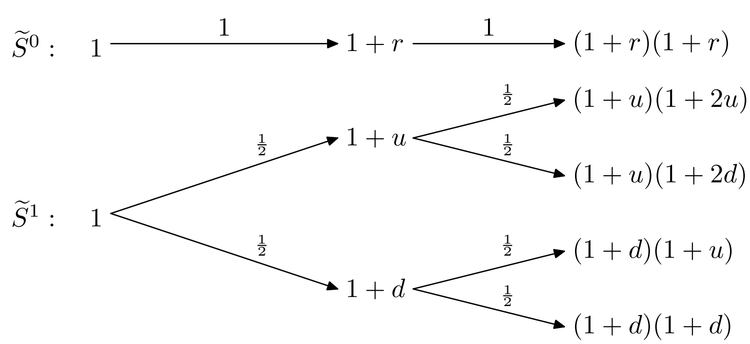
(b) Recall that we call a strategy $\varphi = (\varphi^0, \vartheta)$ self-financing if its discounted cost process $C(\varphi)$ is constant over time. Show that the notion of self-financing strategy does not depend on whether we work with the discounted price processes S^0 and S^1 or the undiscounted processes $\widetilde{S}^0$ and $\widetilde{S}^1$, i.e. show that the discounted cost process $C(\varphi)$ is constant over time if and only if the undiscounted cost process $\widetilde{C}(\varphi)$, determined by

$$\Delta \widetilde{C}(\varphi) := \widetilde{C}_{k+1}(\varphi) - \widetilde{C}_k(\varphi) = (\varphi^0_{k+1} - \varphi^0_k)\widetilde{S}^0_k + (\vartheta_{k+1} - \vartheta_k)\widetilde{S}^1_k,$$

is constant over time.

(c) Use the result in (b) to conclude that the notion of self-financing strategy is numeraire-invariant, i.e. that it does not matter for this definition whether the discounted price processes are defined as $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$, or $\bar{S}^0 := \widetilde{S}^0/\widetilde{S}^1$ and $\bar{S}^1 := \widetilde{S}^1/\widetilde{S}^1$.

Exercise 2.2 Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ given by the following trees, where the numbers beside the branches denote transition probabilities.



Intuitively, this means that the volatility of $\widetilde{S}^1$ increases if the stock price increases in the first period. Assume that u, $r \geq 0$ and $-0.5 < d \leq 0$.

(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables Y_1 and Y_2 and two adapted stochastic processes $\widetilde{S}^0$ and $\widetilde{S}^1$ such that $\widetilde{S}^1_k = \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$.

(b) For which values of u and d are Y_1 and Y_2 *uncorrelated*?

(c) For which values of u and d are Y_1 and Y_2 *independent*?

(d) For which values of u, r and d is the discounted stock process S^1 a P-martingale?

Exercise 2.3 Consider for a finite time horizon $T \geq 2$ a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $\widetilde{S}^1_0 = 1$ and $\widetilde{S}^1_k > 0$ P-a.s. for all $k = 0, \dots, T$. Fix thresholds $0 < \ell < 1 < u$ and define

$$\sigma(\omega) := \inf\{k = 0, \dots, T : S^1_k(\omega) \leq \ell\} \wedge T,$$
$$\tau(\omega) := \inf\{k = \sigma(\omega), \dots, T : S^1_k(\omega) \geq u\} \wedge T,$$

where $\inf \emptyset = +\infty$ as usual. Moreover, for $k = 0, \dots, T$ define

$$\vartheta_k(\omega) := \mathbb{1}_{\{\sigma(\omega) < k \leq \tau(\omega)\}} .$$

Finally define the filtration $\mathbb{F} = (\mathcal{F}_k)_{0 \leq k \leq T}$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_k = \sigma(\widetilde{S}^1_i,\ i \leq k)$.

(a) Show that σ and τ are *stopping times*, i.e. that for all $k = 0, \dots, T$, we have

$$\{\sigma \leq k\}, \{\tau \leq k\} \in \mathcal{F}_k .$$

(b) Show that ϑ is a predictable process with $\vartheta_0 = \vartheta_1 = 0$.

(c) Construct φ^0 such that $\varphi = (\varphi^0, \vartheta)$ is a self-financing strategy with $V_0(\varphi) = 0$ and derive a formula for the (discounted) value process $V(\varphi)$ involving only the discounted stock price S^1 and the stopping times σ and τ.

(d) Describe the trading strategy φ in words.