Mathematical Foundations for Finance

Exercise sheet 4

Please hand in your solutions until Tuesday, 17/10/2017, 18:00 into your assistant's box next to HG G 53.2.

Exercise 4.1 Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Recall that a *stopping time* is a random variable $\tau : \Omega \to \{0, 1, \dots, T\}$ with the property that

$$\{\tau \le k\} \in \mathcal{F}_k$$

for $k = 0, 1, \dots, T$. Recall also the convention that $\inf \emptyset = +\infty$. If $X = (X_k)_{k=0,1,\dots,T}$ is an $\mathbb{F}$-adapted process and $B \in \mathcal{B}(\mathbb{R})$ a Borel set, then

$$\tau_{X,B} := \inf\{k = 0, 1, \dots, T : X_k \in B\}$$

is called the *first hitting time* of X on B.

(a) Show that $\tau_{X,B} \wedge T$ is a stopping time.

(b) Let τ be any stopping time. Show that there exist an adapted process X and a set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$. In other words, show that (up to truncating at T) every (first) hitting time of some adapted process X on some $B \in \mathcal{B}(\mathbb{R})$ is a stopping time and vice versa.
Hint: Try to construct such a process explicitly. It will depend on τ.

Exercise 4.2 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *binomial model* and assume that $T = 1$, $u > r > 0$ and $-1 < d < 0$. For $\widetilde{K} > 0$, define the functions $C(\,\bullet\,, \widetilde{K})$ and $P(\,\bullet\,, \widetilde{K}) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$C(x, \widetilde{K}) := (x - \widetilde{K})^+ := \max(0, x - \widetilde{K}) \quad \text{and} \quad P(x, \widetilde{K}) := (\widetilde{K} - x)^+ := \max(0, \widetilde{K} - x).$$

In financial terms, $C(\,\bullet\,, \widetilde{K})$ is the payoff function of a *European call option with strike* $\widetilde{K}$, and $P(\,\bullet\,, \widetilde{K})$ is the payoff function of a *European put option with strike* $\widetilde{K}$.

(a) Construct a self-financing strategy $\varphi^{C(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{C(\widetilde{K})}, \vartheta^{C(\widetilde{K})})$ such that

$$V_1(\varphi^{C(\widetilde{K})}) = \frac{C(\widetilde{S}_1^1, \widetilde{K})}{1 + r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

(b) Construct a self-financing strategy $\varphi^{P(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{P(\widetilde{K})}, \vartheta^{P(\widetilde{K})})$ such that

$$V_1(\varphi^{P(\widetilde{K})}) = \frac{P(\widetilde{S}_1^1, \widetilde{K})}{1 + r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

(c) Prove the *put-call parity*

$$V_0^{P(\widetilde{K})} + S_0^1 = V_0^{C(\widetilde{K})} + \frac{\widetilde{K}}{1 + r}. \qquad (*)$$

Give an economic interpretation of $(*)$.

(d) Compute $\lim_{\widetilde{K}\to\infty} V_0^{C(\widetilde{K})}$, $\lim_{\widetilde{K}\to 0} V_0^{C(\widetilde{K})}$, $\lim_{\widetilde{K}\to\infty} V_0^{P(\widetilde{K})}$ and $\lim_{\widetilde{K}\to 0} V_0^{P(\widetilde{K})}$. Can you guess the result before doing the computations?

Exercise 4.3 Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ with time horizon $T = 1$ consisting of a bank account and one stock defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $\widetilde{S}_0^0 = \widetilde{S}_0^1 = 1$ and $\widetilde{S}_1^1 = e^Y$, where $Y \sim \mathcal{N}(0,1)$ under P. Finally, assume that $\widetilde{S}_1^0 = e^r$ for a deterministic $r \in (0, 1/2)$ and consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

(a) Consider the map $Q : \mathcal{F} \to \mathbb{R}$ given by $Q[A] := E\left[Z\mathbb{1}_A\right]$, where

$$Z := \exp\left(- \left(\frac{1}{2} - r\right) Y - \frac{(\frac{1}{2} - r)^2}{2} \right).$$

Show that Q is a probability measure equivalent to P.
Hint: You can use that for $X \sim \mathcal{N}(\mu, \sigma^2)$, one has $E\left[e^{\alpha X}\right] = \exp(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2)$.

(b) Show that Q is an equivalent martingale measure for S^1, i.e. that S^1 is a martingale under Q.
Hint: In this setting, $E_Q\left[S_1^1\right] = E\left[ZS_1^1\right]$.

(c) Consider again the (undiscounted) payoff $C(\widetilde{S}_1^1, \widetilde{K}) = (\widetilde{S}_1^1 - \widetilde{K})^+$ of a long position in a European call option with strike $\widetilde{K}$. Compute

$$V_0^C := E_Q\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{\widetilde{S}_1^0}\right].$$

(d) Consider an enlargement of the market given by $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$, where we set $\widetilde{S}_0^2 := V_0^C$ and $\widetilde{S}_1^2 := C(\widetilde{S}_1^1, \widetilde{K})$. Is this market free of arbitrage?