Mathematical Foundations for Finance

Exercise sheet 8

Please hand in your solutions until Tuesday, 14/11/2017, 18:00 into your assistant's box next to HG G 53.2.

Exercise 8.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion (BM) defined on some probability space $(\Omega, \mathcal{F}, P)$ (without filtration). Show that

(a) $W^1 := -W$ is a BM.

(b) $W_t^2 := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha B + \sqrt{1-\alpha^2}B'$ is a BM, where B and B' are two independent BMs and $\alpha \in [0,1]$.

(d) Show that the independence of B and B' in (c) cannot be omitted, i.e., if B and B' are *not* independent, then W^3 need not be a BM. Give two examples.

Exercise 8.2 Let $(\Pi_n)_{n\in\mathbb{N}}$ be a sequence of refining partitions of $[a,b] \subseteq \mathbb{R}$ (in the sense that $\Pi_n \subseteq \Pi_{n+1}$ for all $n \in \mathbb{N}$) with $|\Pi_n| \to 0$ as $n \to \infty$. Let $p > 0$. We define for a function $f : \mathbb{R} \to \mathbb{R}$ its p-variation on $[a,b]$ along the sequence $(\Pi_n)_{n\in\mathbb{N}}$ as

$$V_p^{(a,b)}(f) := \lim_{n\to\infty} \sum_{t_i\in\Pi_n} |f(t_i) - f(t_{i-1})|^p,$$

assuming that the limit exists. Assume additionally that f is continuous on $[a,b]$.

(a) Show that if $V_{p^*}^{(a,b)}(f)$ is finite and non-zero for some $p^* > 0$, then $V_p^{(a,b)}(f) = \infty$ for all $p < p^*$.
Hint: Make sure to use the continuity of f. Use also that every function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on a closed and bounded interval $[a,b]$ is also uniformly continuous on $[a,b]$.

(b) Show that if $V_{p^*}^{(a,b)}(f)$ is finite and non-zero for some $p^* > 0$, then $V_p^{(a,b)}(f) = 0$ for all $p > p^*$.

Exercise 8.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some sufficiently rich filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous convex function. Show that if the stochastic process $\big(f(W_t)\big)_{t\geq 0}$ is integrable, then it is a $(P, \mathbb{F})$-submartingale.
Hint: We have done something similar in discrete time.

(b) Given a $(P, \mathbb{F})$-martingale $(M_t)_{t\geq 0}$ and a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, the process

$$\big(M_t + g(t)\big)_{t\geq 0}$$

is a $(P, \mathbb{F})$-supermartingale if and only if g is decreasing, and a $(P, \mathbb{F})$-submartingale if and only if g is increasing.

(c) Show that the following stochastic processes are $(P, \mathbb{F})$-submartingales but not martingales:

(i) W^2,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: Use the result from (a) and (b), respectively.

(d) Show that any $(P, \mathbb{F})$-local martingale which is null at 0 and uniformly bounded from below is a $(P, \mathbb{F})$-supermartingale.
Hint: We have done this in discrete time already.