ETH Zürich, Fall 2017
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Mathematical Foundations for Finance

Exercise sheet 9

Please hand in your solutions until Tuesday, 21/11/2017, 18:00 into your assistant's box next to HG G 53.2.

Exercise 9.1 Let $(Y_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$. Let $E[Y_k] = \mu$ and $\mathrm{Var}(Y_k) = \sigma^2$ for all $k \in \mathbb{N}$ with $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_+$. Define additionally $X = (X_k)_{k\in\mathbb{N}_0}$ by

$$X_k = \sum_{j=1}^{k} Y_j \quad \text{for all } k \in \mathbb{N}_0,$$

and assume that X is adapted to $\mathbb{F}$.

(a) Show that for any $\mathbb{F}$-adapted integrable process $Z = (Z_k)_{k\in\mathbb{N}_0}$, there exists a P-a.s. unique decomposition of Z into $Z = M + A$ with $M = (M_k)_{k\in\mathbb{N}_0}$ a $(P, \mathbb{F})$-martingale and $A = (A_k)_{k\in\mathbb{N}_0}$ an $\mathbb{F}$-predictable integrable process with $A_0 = 0$.
Hint: This is the Doob decomposition. Show the existence by construction.

(b) Using (a), explicitly derive the processes M and A in the Doob decomposition of X.

(c) Explicitly derive the optional quadratic variation $[M] = \big([M]_k\big)_{k\in\mathbb{N}_0}$ of the square-integrable martingale M from (b), and show that $M^2 - [M]$ is a martingale.
Hint: See Theorem V.1.1 in the lecture notes, and use that due to the condition $\Delta[M] = (\Delta M)^2$, we must have that $[M]_k - [M]_{k-1} = (M_k - M_{k-1})^2$.

(d) Explicitly derive the predictable compensator $\langle M\rangle = \big(\langle M\rangle_k\big)_{k\in\mathbb{N}_0}$ of the process M from (b).
Hint: See the remark on page 79 in the lecture notes. Also use that if M is a square-integrable martingale, then $\langle M\rangle$ is integrable.

Exercise 9.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. Show that the process $X = (X_t)_{t\geq 0}$ defined by

$$X_t := W_t^3 - 3tW_t,$$

is a $(P, \mathbb{F})$-martingale.
Hint: You can use the results from Proposition IV.2.2 in the lecture notes.

Exercise 9.3 A *Poisson process* with parameter $\lambda > 0$ with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a (real-valued) stochastic process $N = (N_t)_{t\geq 0}$ which is adapted to $\mathbb{F}$, $N_0 = 0$ P-a.s. and satisfies the following two properties:

(PP1) For $0 \leq s < t$, the *increment* $N_t - N_s$ is independent (under P) of $\mathcal{F}_s$ and is (under P) *Poisson-distributed* with parameter $\lambda(t-s)$, i.e.

$$P[N_t - N_s = k] = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for P-almost all $\omega \in \Omega$, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant, $\mathbb{N}_0$-valued, and increases by jumps of size 1.

Poisson processes form the cornerstone of *jump processes*, which are of importance in advanced financial modeling. Show that the following processes are $(P, \mathbb{F})$-martingales:

(a) $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$. This process is also called a *compensated Poisson process.*
Hint: If $X \sim Poi(\lambda)$, *then* $E[X] = \lambda$.

(b) $\widetilde{N}_t^2 - N_t$, $t \geq 0$, and $\widetilde{N}_t^2 - \lambda t$, $t \geq 0$. Use these results to derive $[\widetilde{N}]$ and $\langle \widetilde{N} \rangle$.
Hint: If $X \sim Poi(\lambda)$, *then* $\mathrm{Var}(X) = \lambda$.

(c) $S_t := e^{N_t \log(1+\sigma) - \lambda \sigma t}$, $t \geq 0$, where $\sigma > -1$. S is also called a *geometric Poisson process.*