ETH Zürich, D-MATH
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Non-Life Insurance: Mathematics and Statistics

Exercise sheet 1

Exercise 1.1 Discrete Distribution
Suppose the random variable N follows a geometric distribution with parameter $p \in (0,1)$, i.e.

$$\mathbb{P}[N = k] = \begin{cases} (1-p)^{k-1}p & \text{if } k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{else.} \end{cases}$$

(a) Show that the geometric distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) Let $n \in \mathbb{N} \setminus \{0\}$. Calculate $\mathbb{P}[N \geq n]$.

(c) Calculate $\mathbb{E}[N]$.

(d) Let $r < -\log(1-p)$. Calculate $M_N(r) = \mathbb{E}[\exp\{rN\}]$.
Remark: M_N is called the moment generating function of N.

(e) Calculate $\frac{d}{dr} M_N(r)|_{r=0}$. What do you observe?

Exercise 1.2 Absolutely Continuous Distribution
Suppose the random variable Y follows an exponential distribution with parameter $\lambda > 0$, i.e. the density f_Y of Y is given by

$$f_Y(x) = \begin{cases} \lambda \exp\{-\lambda x\} & \text{if } x \geq 0, \\ 0 & \text{else.} \end{cases}$$

(a) Show that the exponential distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) Let $0 < y_1 < y_2$. Calculate $\mathbb{P}[y_1 \leq Y \leq y_2]$.

(c) Calculate $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

(d) Let $r < \lambda$. Calculate $\log M_Y(r) = \log \mathbb{E}[\exp\{rY\}]$.
Remark: $\log M_Y$ is called the cumulant generating function of Y.

(e) Calculate $\frac{d^2}{dr^2} \log M_Y(r)|_{r=0}$. What do you observe?

Exercise 1.3 Conditional Distribution
Suppose that an insurance company distinguishes between small and large claims, where a claim is called large if it exceeds a fixed threshold $\theta > 0$. We use the random variable I to indicate whether a claim is small or large, i.e. we have $I = 0$ for a small claim and $I = 1$ for a large claim. In particular, we get $\mathbb{P}[I = 0] = 1 - p$ and $\mathbb{P}[I = 1] = p$ for some $p \in (0,1)$. Finally, we model the size of a claim that belongs to the large claims section by the random variable Y, where, given that we have a small claim, Y is equal to 0 almost surely and, given that we have a large claim, Y follows a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 0$, i.e. the density $f_{Y|I=1}$ of $Y \mid I = 1$ is given by

$$f_{Y|I=1}(x) = \begin{cases} \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} & \text{if } x \geq \theta, \\ 0 & \text{else.} \end{cases}$$

(a) Let $y > \theta$. Calculate $\mathbb{P}[Y \geq y]$.

(b) Calculate $\mathbb{E}[Y]$.