Exercise 12.1 Chain-Ladder and Bornhuetter-Ferguson

Set $I = J + 1 = 10$. Assume that for the cumulative payments $C_{i,j}$ with $1 \leq i \leq 10$ and $0 \leq j \leq 9$ we have observations $D_{10} = \{ C_{i,j} \mid i + j \leq 10, 1 \leq i \leq 10, 0 \leq j \leq 9 \}$
given by the following claims development triangle:

<table>
<thead>
<tr>
<th>accident year $i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5'946'975</td>
<td>9'668'212</td>
<td>10'771'690</td>
<td>10'978'394</td>
<td>11'040'518</td>
<td>11'106'331</td>
<td>11'121'181</td>
<td>11'132'310</td>
<td>11'148'124</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6'346'756</td>
<td>9'593'162</td>
<td>10'316'383</td>
<td>10'468'180</td>
<td>10'536'004</td>
<td>10'572'608</td>
<td>10'625'360</td>
<td>10'636'546</td>
<td>10'648'192</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6'269'090</td>
<td>9'245'313</td>
<td>10'092'366</td>
<td>10'355'134</td>
<td>10'507'837</td>
<td>10'573'282</td>
<td>10'626'827</td>
<td>10'635'751</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5'863'015</td>
<td>8'546'239</td>
<td>9'268'771</td>
<td>9'459'424</td>
<td>9'592'399</td>
<td>9'680'740</td>
<td>9'724'068</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5'778'885</td>
<td>8'524'114</td>
<td>9'178'009</td>
<td>9'451'404</td>
<td>9'681'692</td>
<td>9'786'916</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5'600'184</td>
<td>8'493'391</td>
<td>9'056'505</td>
<td>9'282'022</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5'288'066</td>
<td>7'728'169</td>
<td>8'256'211</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>5'290'793</td>
<td>7'648'729</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>5'675'568</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that you can download this data set from 
https://people.math.ethz.ch/~wueth/exercises2.html
by clicking on “Data to the Examples”.

(a) Predict the lower triangle $D'_{10} = \{ C_{i,j} \mid i + j > 10, 1 \leq i \leq 10, 0 \leq j \leq 9 \}$
using the chain-ladder (CL) method. Calculate the CL reserves $\hat{R}_{CL}^i$ for all accident years $i \in \{1, \ldots, 10\}$.

(b) Assume we have prior informations $\hat{\mu}_1, \ldots, \hat{\mu}_{10}$ for the total expected ultimate claims $E[C_{1,j}], \ldots, E[C_{10,j}]$ given by

<table>
<thead>
<tr>
<th>accident year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>prior information</td>
<td>11'653'101</td>
<td>11'367'306</td>
<td>10'962'965</td>
<td>10'616'762</td>
<td>11'044'881</td>
</tr>
<tr>
<td>accident year</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>prior information</td>
<td>11'480'700</td>
<td>11'413'572</td>
<td>11'126'527</td>
<td>10'986'548</td>
<td>11'618'437</td>
</tr>
</tbody>
</table>

Calculate the claim reserves $\hat{R}_{BF}^i$ for all accident years $i \in \{1, \ldots, 10\}$ using the Bornhuetter-Ferguson (BF) method.

(c) Explain why in this example we have

$\hat{R}_{CL}^i < \hat{R}_{BF}^i$,

for all accident years $i \in \{2, \ldots, 10\}$. 
Exercise 12.2 Mack’s Formula and Merz-Wüthrich (MW) Formula (R Exercise)
Consider the identical setup as in Exercise 12.1. Write an R-Code using the R-package ChainLadder in order to calculate the conditional mean square error of prediction
\[
msep_{C_{i,j}|D_i}(\hat{C}_{i,j}),
\]
for all \(i \in \{1, \ldots, 10\}\), given in formula (9.21) of the lecture notes, as well as
\[
msep_{\sum_{i=1}^{I} C_{i,j}|D_i}(\sum_{i=1}^{I} \hat{C}_{i,j}),
\]
for aggregated accident years, given in formula (9.22) of the lecture notes. Interpret the square-rooted conditional mean square errors of prediction relative to the claims reserves calculated in Exercise 12.1, (a). Moreover, determine
\[
msep_{MW, CDR_{i,t+1}|D_i}(0),
\]
for all \(i \in \{1, \ldots, 10\}\), given in formula (9.34) of the lecture notes as well as
\[
msep_{\sum_{i=1}^{I} CDR_{i,t+1}|D_i}(0),
\]
for aggregated accident years, given in formula (9.35) of the lecture notes.

Exercise 12.3 Conditional MSEP and Claims Development Result
In the first part of this exercise we show that the conditional mean square error of prediction can be decoupled into the process uncertainty and the parameter estimation error. In the second part we show that the claims development results are uncorrelated.

(a) Let \(\mathcal{D}\) be a \(\sigma\)-algebra, \(\hat{X}\) a \(\mathcal{D}\)-measurable predictor for the random variable \(X\) and both \(\hat{X}\) and \(X\) square-integrable. Show that
\[
msep_{X|\mathcal{D}}(\hat{X}) = \mathbb{E} \left[ (X - \hat{X})^2 \bigg| \mathcal{D} \right] = \text{Var}(X|\mathcal{D}) + \left( \mathbb{E}[X|\mathcal{D}] - \hat{X} \right)^2 \quad \text{a.s.}
\]

(b) Suppose that we are in the framework of Chapter 9.4.1 of the lecture notes. Let \(t_1 < t_2 \in \mathbb{N}\) such that \(t_1 \geq I\) and \(i > t_2 - J\) and assume that \(C_{i,j}\) has finite second moment. Show that
\[
\text{Cov}(\text{CDR}_{i,t_1+1}, \text{CDR}_{i,t_2+1}) = 0.
\]