Non-Life Insurance: Mathematics and Statistics

Exercise sheet 2

Exercise 2.1 Gaussian Distribution

For a random variable X we write $X \sim \mathcal{N}(\mu, \sigma^2)$ if X follows a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. The moment generating function M_X of $X \sim \mathcal{N}(\mu, \sigma^2)$ is given by

$$M_X(r) = \exp\left\{ r\mu + \frac{r^2\sigma^2}{2} \right\} \qquad \text{for all } r \in \mathbb{R}.$$

(a) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $a, b \in \mathbb{R}$. Show that

$$a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2).$$

(b) Let $X_1, \ldots, X_n$ be independent with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for all $i \in \{1, \ldots, n\}$. Show that

$$\sum_{i=1}^n X_i \sim \mathcal{N}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2 \right).$$

Exercise 2.2 Maximum Likelihood and Hypothesis Test

Let $Y_1, \ldots, Y_n$ be claim amounts in CHF that an insurance company has to pay. We assume that $Y_1, \ldots, Y_n$ are independent random variables that all follow a log-normal distribution with the same unknown parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Then, by definition, $\log Y_1, \ldots, \log Y_n$ are independent Gaussian random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Let $n = 8$ and suppose that we have the following observations for $\log Y_1, \ldots, \log Y_8$:

$$x_1 = 9, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 7, \quad x_5 = 3, \quad x_6 = 11, \quad x_7 = 6, \quad x_8 = 10.$$

(a) Write down the joint density $f_{\mu,\sigma^2}(x_1, \ldots, x_8)$ of $\log Y_1, \ldots, \log Y_8$.

(b) Calculate $\log f_{\mu,\sigma^2}(x_1, \ldots, x_8)$.

(c) Calculate $(\hat{\mu}, \hat{\sigma}^2) = \arg \max_{(\mu,\sigma^2) \in \mathbb{R} \times \mathbb{R}_{>0}} \log f_{\mu,\sigma^2}(x_1, \ldots, x_8)$.

Remark: Since the logarithm is a monotonically increasing function, $\hat{\mu}$ and $\hat{\sigma}^2$ are chosen such that the joint density of $\log Y_1, \ldots, Y_n$ at the given observations $x_1, \ldots, x_8$ is maximized. Hence $\hat{\mu}$ and $\hat{\sigma}^2$ are called maximum likelihood estimates.

(d) Now suppose that we are interested in the mean μ of the logarithm of the claim amounts and an expert claims that $\mu = 6$. Perform a statistical test to test the null hypothesis

$$H_0 : \mu = 6$$

against the (two-sided) alternative hypothesis

$$H_1 : \mu \neq 6.$$

Exercise 2.3 Variance Decomposition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ any sub-σ-algebra of $\mathcal{F}$. Show that

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}(\mathbb{E}[X|\mathcal{G}]).$$