

# Non-Life Insurance: Mathematics and Statistics

## Exercise sheet 5

### Exercise 5.1 Kolmogorov-Smirnov Test

Suppose we are given the following claim size data (in increasing order) coming from independent realizations of an unknown claim size distribution:

$$x_1 = \left(-\log \frac{38}{40}\right)^2, x_2 = \left(-\log \frac{37}{40}\right)^2, x_3 = \left(-\log \frac{35}{40}\right)^2, x_4 = \left(-\log \frac{34}{40}\right)^2, x_5 = \left(-\log \frac{10}{40}\right)^2.$$

Perform a Kolmogorov-Smirnov test at significance level of 5% to test the null hypothesis of having a Weibull distribution with shape parameter  $\tau = \frac{1}{2}$  and scale parameter  $c = 1$  as claim size distribution. Moreover, explain why the Kolmogorov-Smirnov test is applicable in this example.

### Exercise 5.2 Large Claims

We define storm and flood events to be those events with total claim amount exceeding CHF 50 millions. Assume that for the yearly claim amount  $S$  of such storm and flood events we have  $S \sim \text{CompPoi}(\lambda, G)$ , where  $\lambda$  is unknown and  $G$  is the distribution function of a Pareto distribution with threshold  $\theta = 50$  and unknown tail index  $\alpha > 0$ . Moreover, the number of claims and the claim sizes and thus also the total claim amounts are assumed to be independent across different years. During the years 1986 – 2005 we observed the following 15 storm and flood events:

date	amount in millions	date	amount in millions
20.06.1986	52.8	18.05.1994	78.5
18.08.1986	135.2	18.02.1999	75.3
18.07.1987	55.9	12.05.1999	178.3
23.08.1987	138.6	26.12.1999	182.8
26.02.1990	122.9	04.07.2000	54.4
21.08.1992	55.8	13.10.2000	365.3
24.09.1993	368.2	20.08.2005	1'051.1
08.10.1993	83.8		

- (a) Using  $n$  independent claim sizes  $Y_1, \dots, Y_n$ , show that the MLE  $\hat{\alpha}_n^{\text{MLE}}$  of  $\alpha$  is given by

$$\hat{\alpha}_n^{\text{MLE}} = \left( \frac{1}{n} \sum_{i=1}^n \log Y_i - \log \theta \right)^{-1}.$$

- (b) According to Lemma 3.7 of the lecture notes,

$$\frac{n-1}{n} \hat{\alpha}_n^{\text{MLE}}$$

is an unbiased version of the MLE. Estimate  $\alpha$  using the unbiased version of the MLE for the storm and flood data given above.

- (c) Calculate the MLE of  $\lambda$  for the storm and flood data given above.

- (d) Suppose that we introduce a maximal claims cover of  $M = 2$  billions CHF per event, i.e. the individual claims are given by  $\min\{Y_i, M\}$ . Using the estimates of  $\alpha$  and  $\lambda$  found above, calculate the estimated expected total yearly claim amount.
- (e) Using the estimates of  $\alpha$  and  $\lambda$  found above, calculate the estimated probability that we observe at least one storm and flood event next year which exceeds the level of  $M = 2$  billions CHF.

### Exercise 5.3 Pareto Distribution

Suppose the random variable  $Y$  follows a Pareto distribution with threshold  $\theta > 0$  and tail index  $\alpha > 0$ .

- (a) Show that the survival function of  $Y$  is regularly varying at infinity with tail index  $\alpha$ .
- (b) Show that for  $\theta \leq u_1 < u_2$  the expected value of  $Y$  within the layer  $(u_1, u_2]$  is given by

$$\mathbb{E}[Y 1_{\{u_1 < Y \leq u_2\}}] = \begin{cases} \theta \frac{\alpha}{\alpha-1} \left[ \left(\frac{u_1}{\theta}\right)^{-\alpha+1} - \left(\frac{u_2}{\theta}\right)^{-\alpha+1} \right] & \text{if } \alpha \neq 1, \\ \theta \log\left(\frac{u_2}{u_1}\right) & \text{if } \alpha = 1. \end{cases}$$

- (c) Show that for  $\alpha > 1$  and  $y > \theta$  the loss size index function for level  $y$  is given by

$$\mathcal{I}[G(y)] = 1 - \left(\frac{y}{\theta}\right)^{-\alpha+1}.$$

- (d) Show that for  $\alpha > 1$  and  $u > \theta$ , the mean excess function of  $Y$  above  $u$  is given by

$$e(u) = \frac{1}{\alpha-1}u.$$