Non-Life Insurance: Mathematics and Statistics

Solution sheet 1

Solution 1.1 Discrete Distribution

(a) Note that $N$ only takes values in $\mathbb{N} \setminus \{0\}$ and that $p \in (0, 1)$. Hence we calculate

$$P[N \in \mathbb{R}] = \sum_{k=1}^{\infty} P[N = k] = \sum_{k=1}^{\infty} (1 - p)^{k-1}p = p \sum_{k=0}^{\infty} (1 - p)^k = \frac{p}{1 - (1 - p)} = \frac{1}{p} = 1,$$

from which we can conclude that the geometric distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) For $n \in \mathbb{N} \setminus \{0\}$, we get

$$P[N \geq n] = \sum_{k=n}^{\infty} P[N = k] = \sum_{k=n}^{\infty} (1 - p)^{k-1}p = (1 - p)^{n-1}p \sum_{k=0}^{\infty} (1 - p)^k = (1 - p)^{n-1},$$

where we used that $\sum_{k=0}^{\infty} (1 - p)^k = \frac{1}{p}$, as was shown in (a).

(c) The expectation of a discrete random variable that takes values in $\mathbb{N} \setminus \{0\}$ can be calculated as

$$E[N] = \sum_{k=1}^{\infty} k \cdot P[N = k].$$

Thus we get

$$E[N] = \sum_{k=1}^{\infty} k (1 - p)^{k-1}p = \sum_{k=0}^{\infty} (k+1)(1 - p)^k p = \sum_{k=0}^{\infty} k (1 - p)^k p + \sum_{k=0}^{\infty} (1 - p)^k p = (1 - p)E[N] + 1,$$

where we used that $\sum_{k=0}^{\infty} (1 - p)^k p = 1$, as was shown in (a). We conclude that $E[N] = \frac{1}{p}$.

(d) Let $r \in \mathbb{R}$. Then we calculate

$$E[\exp(rN)] = \sum_{k=1}^{\infty} \exp(rk) \cdot P[N = k]$$

$$= \sum_{k=1}^{\infty} \exp(rk) (1 - p)^{k-1}p$$

$$= p \sum_{k=1}^{\infty} [(1 - p) \exp(r)]^{k-1}$$

$$= p \sum_{k=0}^{\infty} [(1 - p) \exp(r)]^k.$$

Since $(1 - p) \exp(r)$ is strictly positive, the sum on the right hand side is convergent if and only if $(1 - p) \exp(r) < 1$, which is equivalent to $r < -\log(1 - p)$. Hence $E[\exp(rN)]$ exists if and only if $r < -\log(1 - p)$ and in this case we have

$$M_N(r) = E[\exp(rN)] = p \exp(r) \frac{1}{1 - (1 - p) \exp(r)} = \frac{p \exp(r)}{1 - (1 - p) \exp(r)}.$$
(e) For $r < -\log(1 - p)$, we have
\[
\frac{d}{dr} M_N(r) = \frac{p \exp\{r\}}{1 - (1 - p) \exp\{r\}} \\
= \frac{p \exp\{r\} [1 - (1 - p) \exp\{r\}] + p \exp\{r\} (1 - p) \exp\{r\}}{[1 - (1 - p) \exp\{r\}]^2} \\
= \frac{p \exp\{r\}}{[1 - (1 - p) \exp\{r\}]^2}.
\]

Hence we get
\[
\left. \frac{d}{dr} M_N(r) \right|_{r = 0} = \frac{p \exp\{0\}}{[1 - (1 - p) \exp\{0\}]^2} = \frac{p}{p^2} = \frac{1}{p}.
\]

We observe that $\left. \frac{d}{dr} M_N(r) \right|_{r = 0} = \mathbb{E}[N]$, which holds in general for all random variables if the moment generating function exists in an interval around 0.

### Solution 1.2 Absolutely Continuous Distribution

(a) We calculate
\[
P[Y \in \mathbb{R}] = \int_{-\infty}^{\infty} f_Y(x) \, dx = \int_{0}^{\infty} \lambda \exp\{-\lambda x\} \, dx = [\exp\{-\lambda x\}]_{0}^{\infty} = [-0 - (-1)] = 1,
\]
from which we can conclude that the exponential distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) For $0 < y_1 < y_2$, we calculate
\[
P[y_1 \leq Y \leq y_2] = \int_{y_1}^{y_2} f_Y(x) \, dx \\
= \int_{y_1}^{y_2} \lambda \exp\{-\lambda x\} \, dx \\
= [\exp\{-\lambda x\}]_{y_1}^{y_2} \\
= \exp\{-\lambda y_2\} - \exp\{-\lambda y_1\}.
\]

(c) The expectation and the second moment of an absolutely continuous random variable can be calculated as
\[
\mathbb{E}[Y] = \int_{-\infty}^{\infty} x \, f_Y(x) \, dx \quad \text{and} \quad \mathbb{E}[Y^2] = \int_{-\infty}^{\infty} x^2 \, f_Y(x) \, dx.
\]

Thus, using partial integration, we get
\[
\mathbb{E}[Y] = \int_{0}^{\infty} x \lambda \exp\{-\lambda x\} \, dx \\
= [-x \exp\{-\lambda x\}]_{0}^{\infty} + \int_{0}^{\infty} \exp\{-\lambda x\} \, dx \\
= 0 + \left[ -\frac{1}{\lambda} \exp\{-\lambda x\} \right]_{0}^{\infty} \\
= \frac{1}{\lambda}.
\]

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The variance \( \text{Var}(Y) \) can be calculated as

\[
\text{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{E}[Y^2] - \frac{1}{\lambda^2}.
\]

For the second moment \( \mathbb{E}[Y^2] \) we get, again using partial integration,

\[
\mathbb{E}[Y^2] = \int_0^{\infty} x^2 \lambda \exp(-\lambda x) \, dx = \left[ -x^2 \exp(-\lambda x) \right]_0^\infty + \int_0^{\infty} 2x \exp(-\lambda x) \, dx = 0 + \frac{2}{\lambda} \mathbb{E}[Y] = \frac{2}{\lambda^2};
\]

from which we can conclude that

\[
\text{Var}(Y) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.
\]

Note that for the exponential distribution both the expectation and the variance exist. The reason is that \( \exp(-\lambda x) \) goes much faster to 0 than \( x \) or \( x^2 \) go to infinity, for all \( \lambda > 0 \).

(d) Let \( r \in \mathbb{R} \). Then we calculate

\[
\mathbb{E}[\exp(rY)] = \int_0^{\infty} \exp(rx) \lambda \exp(-\lambda x) \, dx = \int_0^{\infty} \lambda \exp\{(r-\lambda)x\} \, dx.
\]

The integral on the right hand side and therefore also \( \mathbb{E}[\exp(rY)] \) exist if and only if \( r < \lambda \). In this case we have

\[
M_Y(r) = \mathbb{E}[\exp(rY)] = \frac{\lambda}{r-\lambda} \left[ \exp\{(r-\lambda)x\} \right]_0^{\infty} = \frac{\lambda}{r-\lambda} (0 - 1) = \frac{\lambda}{\lambda - r}
\]

and therefore

\[
\log M_Y(r) = \log \left( \frac{\lambda}{\lambda - r} \right).
\]

(e) For \( r < \lambda \), we have

\[
\frac{d^2}{dr^2} \log M_Y(r) = \frac{d^2}{dr^2} \log \left( \frac{\lambda}{\lambda - r} \right) = \frac{d^2}{dr^2} \left[ \log(\lambda) - \log(\lambda - r) \right] = \frac{d}{dr} \frac{1}{\lambda - r} = \frac{1}{(\lambda - r)^2}.
\]

Hence we get

\[
\frac{d^2}{dr^2} \log M_Y(r)|_{r=0} = \frac{1}{(\lambda - 0)^2} = \frac{1}{\lambda^2}.
\]

We observe that \( \frac{d^2}{dr^2} \log M_Y(r)|_{r=0} = \text{Var}(Y) \), which holds in general for all random variables if the moment generating function exists in an interval around 0.

**Solution 1.3 Conditional Distribution**

(a) For \( y > \theta > 0 \), we get

\[
P[Y \geq y] = P[Y \geq y, I = 0] + P[Y \geq y, I = 1]
= P[Y \geq y|I = 0]P[I = 0] + P[Y \geq y|I = 1]P[I = 1]
= 0 \cdot (1 - p) + P[Y \geq y|I = 1] \cdot p
= p \cdot P[Y \geq y|I = 1],
\]
since $Y|I = 0$ is equal to 0 almost surely and thus $P[Y \geq y|I = 0] = 0$. Since $Y | I = 1 \sim \text{Pareto}(\theta, \alpha)$, we can calculate

$$P[Y \geq y|I = 1] = \int_y^\infty f_{Y|I=1}(x) \, dx = \int_y^\infty \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} \, dx = \left[-\left(\frac{x}{\theta}\right)^{-\alpha}\right]_y^\infty = \left(\frac{y}{\theta}\right)^{-\alpha}.$$ 

We conclude that

$$P[Y \geq y] = p \left(\frac{y}{\theta}\right)^{-\alpha}.$$

(b) Using that $Y|I = 0$ is equal to 0 almost surely and thus $E[Y|I = 0] = 0$, we get

$$E[Y] = E[Y \cdot 1_{I=0}] + E[Y \cdot 1_{I=1}] = E[Y|I = 0]P[I = 0] + E[Y|I = 1]P[I = 1] = p \cdot E[Y|I = 1].$$

Since $Y | I = 1 \sim \text{Pareto}(\theta, \alpha)$, we can calculate

$$E[Y|I = 1] = \int_{-\infty}^{\infty} x f_{Y|I=1}(x) \, dx = \int_{\theta}^{\infty} x \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} \, dx = \alpha \theta \int_{\theta}^{\infty} x^{-\alpha} \, dx$$

We see that the integral on the right hand side and therefore also $E[Y]$ exist if and only if $\alpha > 1$. In this case we get

$$E[Y|I = 1] = \alpha \theta \left[-\frac{1}{\alpha-1} x^{-(\alpha-1)}\right]_\theta^{\infty} = \alpha \theta \frac{1}{\alpha-1} \theta^{-(\alpha-1)} = \theta \frac{\alpha}{\alpha - 1}.$$

We conclude that, if $\alpha > 1$, we get

$$E[Y] = p \theta \frac{\alpha}{\alpha - 1}.$$

If $0 < \alpha \leq 1$, $E[Y]$ does not exist.