Non-Life Insurance: Mathematics and Statistics

Solution sheet 1

Solution 1.1 Discrete Distribution

(a) Note that N only takes values in $\mathbb{N} \setminus \{0\}$ and that $p \in (0,1)$. Hence we calculate

$$\mathbb{P}[N \in \mathbb{R}] = \sum_{k=1}^{\infty} \mathbb{P}[N = k] = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{k=0}^{\infty} (1-p)^k = p \frac{1}{1-(1-p)} = p \frac{1}{p} = 1,$$

from which we can conclude that the geometric distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) For $n \in \mathbb{N} \setminus \{0\}$, we get

$$\mathbb{P}[N \geq n] = \sum_{k=n}^{\infty} \mathbb{P}[N = k] = \sum_{k=n}^{\infty} (1-p)^{k-1} p = (1-p)^{n-1} p \sum_{k=0}^{\infty} (1-p)^k = (1-p)^{n-1},$$

where we used that $\sum_{k=0}^{\infty} (1-p)^k = \frac{1}{p}$, as was shown in (a).

(c) The expectation of a discrete random variable that takes values in $\mathbb{N} \setminus \{0\}$ can be calculated as

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} k \cdot \mathbb{P}[N = k].$$

Thus we get

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = \sum_{k=0}^{\infty} (k+1)(1-p)^k p = \sum_{k=0}^{\infty} k(1-p)^k p + \sum_{k=0}^{\infty} (1-p)^k p = (1-p)\mathbb{E}[N] + 1,$$

where we used that $\sum_{k=0}^{\infty} (1-p)^k p = 1$, as was shown in (a). We conclude that $\mathbb{E}[N] = \frac{1}{p}$.

(d) Let $r \in \mathbb{R}$. Then we calculate

$$\begin{aligned}
\mathbb{E}[\exp\{rN\}] &= \sum_{k=1}^{\infty} \exp\{rk\} \cdot \mathbb{P}[N = k] \\
&= \sum_{k=1}^{\infty} \exp\{rk\} (1-p)^{k-1} p \\
&= p \exp\{r\} \sum_{k=1}^{\infty} [(1-p)\exp\{r\}]^{k-1} \\
&= p \exp\{r\} \sum_{k=0}^{\infty} [(1-p)\exp\{r\}]^{k}.
\end{aligned}$$

Since $(1-p)\exp\{r\}$ is strictly positive, the sum on the right hand side is convergent if and only if $(1-p)\exp\{r\} < 1$, which is equivalent to $r < -\log(1-p)$. Hence $\mathbb{E}[\exp\{rN\}]$ exists if and only if $r < -\log(1-p)$ and in this case we have

$$M_N(r) = \mathbb{E}[\exp\{rN\}] = p \exp\{r\} \frac{1}{1-(1-p)\exp\{r\}} = \frac{p \exp\{r\}}{1-(1-p)\exp\{r\}}.$$

(e) For $r < -\log(1-p)$, we have

$$\begin{aligned}\frac{d}{dr}M_N(r) &= \frac{d}{dr}\frac{p\exp\{r\}}{1-(1-p)\exp\{r\}}\\ &= \frac{p\exp\{r\}[1-(1-p)\exp\{r\}] + p\exp\{r\}(1-p)\exp\{r\}}{[1-(1-p)\exp\{r\}]^2}\\ &= \frac{p\exp\{r\}}{[1-(1-p)\exp\{r\}]^2}.\end{aligned}$$

Hence we get

$$\frac{d}{dr}M_N(r)|_{r=0} = \frac{p\exp\{0\}}{[1-(1-p)\exp\{0\}]^2} = \frac{p}{[1-(1-p)]^2} = \frac{p}{p^2} = \frac{1}{p}.$$

We observe that $\frac{d}{dr}M_N(r)|_{r=0} = \mathbb{E}[N]$, which holds in general for all random variables if the moment generating function exists in an interval around 0.

Solution 1.2 Absolutely Continuous Distribution

(a) We calculate

$$\mathbb{P}[Y \in \mathbb{R}] = \int_{-\infty}^{\infty} f_Y(x)\,dx = \int_0^{\infty} \lambda\exp\{-\lambda x\}\,dx = [-\exp\{-\lambda x\}]_0^{\infty} = [-0-(-1)] = 1,$$

from which we can conclude that the exponential distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) For $0 < y_1 < y_2$, we calculate

$$\begin{aligned}\mathbb{P}[y_1 \le Y \le y_2] &= \int_{y_1}^{y_2} f_Y(x)\,dx\\ &= \int_{y_1}^{y_2} \lambda\exp\{-\lambda x\}\,dx\\ &= [-\exp\{-\lambda x\}]_{y_1}^{y_2}\\ &= \exp\{-\lambda y_1\} - \exp\{-\lambda y_2\}.\end{aligned}$$

(c) The expectation and the second moment of an absolutely continuous random variable can be calculated as

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} x f_Y(x)\,dx \qquad \text{and} \qquad \mathbb{E}[Y^2] = \int_{-\infty}^{\infty} x^2 f_Y(x)\,dx.$$

Thus, using partial integration, we get

$$\begin{aligned}\mathbb{E}[Y] &= \int_0^{\infty} x\lambda\exp\{-\lambda x\}\,dx\\ &= [-x\exp\{-\lambda x\}]_0^{\infty} + \int_0^{\infty}\exp\{-\lambda x\}\,dx\\ &= 0 + \left[-\frac{1}{\lambda}\exp\{-\lambda x\}\right]_0^{\infty}\\ &= \frac{1}{\lambda}.\end{aligned}$$

The variance $\mathrm{Var}(Y)$ can be calculated as

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{E}[Y^2] - \frac{1}{\lambda^2}.$$

For the second moment $\mathbb{E}[Y^2]$ we get, again using partial integration,

$$\begin{aligned}\mathbb{E}[Y^2] &= \int_0^\infty x^2 \lambda \exp\{-\lambda x\}\, dx \\ &= \left[-x^2 \exp\{-\lambda x\}\right]_0^\infty + \int_0^\infty 2x \exp\{-\lambda x\}\, dx \\ &= 0 + \frac{2}{\lambda}\mathbb{E}[Y] \\ &= \frac{2}{\lambda^2},\end{aligned}$$

from which we can conclude that

$$\mathrm{Var}(Y) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Note that for the exponential distribution both the expectation and the variance exist. The reason is that $\exp\{-\lambda x\}$ goes much faster to 0 than x or x^2 go to infinity, for all $\lambda > 0$.

(d) Let $r \in \mathbb{R}$. Then we calculate

$$\mathbb{E}[\exp\{rY\}] = \int_0^\infty \exp\{rx\}\lambda \exp\{-\lambda x\}\, dx = \int_0^\infty \lambda \exp\{(r-\lambda)x\}\, dx.$$

The integral on the right hand side and therefore also $\mathbb{E}[\exp\{rY\}]$ exist if and only if $r < \lambda$. In this case we have

$$M_Y(r) = \mathbb{E}[\exp\{rY\}] = \frac{\lambda}{r-\lambda}\left[\exp\{(r-\lambda)x\}\right]_0^\infty = \frac{\lambda}{r-\lambda}(0-1) = \frac{\lambda}{\lambda - r}$$

and therefore

$$\log M_Y(r) = \log\left(\frac{\lambda}{\lambda - r}\right).$$

(e) For $r < \lambda$, we have

$$\frac{d^2}{dr^2}\log M_Y(r) = \frac{d^2}{dr^2}\log\left(\frac{\lambda}{\lambda - r}\right) = \frac{d^2}{dr^2}[\log(\lambda) - \log(\lambda - r)] = \frac{d}{dr}\frac{1}{\lambda - r} = \frac{1}{(\lambda - r)^2}.$$

Hence we get

$$\frac{d^2}{dr^2}\log M_Y(r)|_{r=0} = \frac{1}{(\lambda - 0)^2} = \frac{1}{\lambda^2}.$$

We observe that $\frac{d^2}{dr^2}\log M_Y(r)|_{r=0} = \mathrm{Var}(Y)$, which holds in general for all random variables if the moment generating function exists in an interval around 0.

Solution 1.3 Conditional Distribution

(a) For $y > \theta > 0$, we get

$$\begin{aligned}\mathbb{P}[Y \geq y] &= \mathbb{P}[Y \geq y, I = 0] + \mathbb{P}[Y \geq y, I = 1] \\ &= \mathbb{P}[Y \geq y | I = 0]\mathbb{P}[I = 0] + \mathbb{P}[Y \geq y | I = 1]\mathbb{P}[I = 1] \\ &= 0 \cdot (1-p) + \mathbb{P}[Y \geq y | I = 1] \cdot p \\ &= p \cdot \mathbb{P}[Y \geq y | I = 1],\end{aligned}$$

since $Y|I = 0$ is equal to 0 almost surely and thus $\mathbb{P}[Y \geq y|I = 0] = 0$. Since $Y \mid I = 1 \sim$ Pareto(θ, α), we can calculate

$$\mathbb{P}[Y \geq y|I = 1] = \int_y^\infty f_{Y|I=1}(x)\,dx = \int_y^\infty \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{-(\alpha+1)} dx = \left[-\left(\frac{x}{\theta}\right)^{-\alpha}\right]_y^\infty = \left(\frac{y}{\theta}\right)^{-\alpha}.$$

We conclude that

$$\mathbb{P}[Y \geq y] = p\left(\frac{y}{\theta}\right)^{-\alpha}.$$

(b) Using that $Y|I = 0$ is equal to 0 almost surely and thus $\mathbb{E}[Y|I = 0] = 0$, we get

$$\mathbb{E}[Y] = \mathbb{E}[Y \cdot 1_{\{I=0\}}] + \mathbb{E}[Y \cdot 1_{\{I=1\}}] = \mathbb{E}[Y|I = 0]\mathbb{P}[I = 0] + \mathbb{E}[Y|I = 1]\mathbb{P}[I = 1] = p \cdot \mathbb{E}[Y|I = 1].$$

Since $Y \mid I = 1 \sim$ Pareto(θ, α), we can calculate

$$\mathbb{E}[Y|I = 1] = \int_{-\infty}^\infty x f_{Y|I=1}(x)\,dx = \int_\theta^\infty x\frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{-(\alpha+1)} dx = \alpha\theta^\alpha \int_\theta^\infty x^{-\alpha}\,dx$$

We see that the integral on the right hand side and therefore also $\mathbb{E}[Y]$ exist if and only if $\alpha > 1$. In this case we get

$$\mathbb{E}[Y|I = 1] = \alpha\theta^\alpha\left[-\frac{1}{\alpha - 1}x^{-(\alpha-1)}\right]_\theta^\infty = \alpha\theta^\alpha\frac{1}{\alpha - 1}\theta^{-(\alpha-1)} = \theta\frac{\alpha}{\alpha - 1}.$$

We conclude that, if $\alpha > 1$, we get

$$\mathbb{E}[Y] = p\theta\frac{\alpha}{\alpha - 1}.$$

If $0 < \alpha \leq 1$, $\mathbb{E}[Y]$ does not exist.