Solution 3.1 No-Claims Bonus

(a) We define the following events:

\[ A = \{ \text{“no claims in the last six years”} \}, \]
\[ B = \{ \text{“no claims in the last three years but at least one claim in the last six years”} \}, \]
\[ C = \{ \text{“at least one claim in the last three years”} \}. \]

Note that since the events \( A \), \( B \) and \( C \) are disjoint and cover all possible outcomes, we have

\[ P(A) + P(B) + P(C) = 1, \]

i.e. it is sufficient to calculate two out of the three probabilities. Since the calculation of \( P(B) \) is slightly more involved, we will look at \( P(A) \) and \( P(C) \). Let \( N_1, \ldots, N_6 \) be the number of claims of the last six years of our considered car driver, where \( N_6 \) corresponds to the most recent year. By assumption, \( N_1, \ldots, N_6 \) are independent Poisson random variables with frequency parameter \( \lambda = 0.2 \). Therefore, we can calculate

\[ P(A) = P[N_1 = 0, \ldots, N_6 = 0] = \prod_{i=1}^{6} P[N_i = 0] = \prod_{i=1}^{6} \exp{-\lambda} = \exp{-6\lambda} = \exp{-1.2} \]

and, similarly,

\[ P[C] = 1 - P[C^c] = 1 - P[N_4 = 0, N_5 = 0, N_6 = 0] = 1 - \exp{-3\lambda} = 1 - \exp{-0.6}. \]

For the event \( B \) we get

\[ P(B) = 1 - P[A] - P[C] = 1 - \exp{-1.2} - (1 - \exp{-0.6}) = \exp{-0.6} - \exp{-1.2}. \]

Thus the expected proportion \( q \) of the base premium that is still paid after the grant of the no-claims bonus is given by

\[
q = 0.8 \cdot P[A] + 0.9 \cdot P[B] + 1 \cdot P[C] \\
= 0.8 \cdot \exp{-1.2} + 0.9 \cdot (\exp{-0.6} - \exp{-1.2}) + 1 - \exp{-0.6} \\
\approx 0.915.
\]

If \( s \) denotes the surcharge on the base premium, then it has to satisfy the equation

\[ q(1 + s) \cdot \text{base premium} = \text{base premium}, \]

which leads to

\[ s = \frac{1}{q} - 1. \]

We conclude that the surcharge on the base premium is given by approximately 9.3\%.
(b) We use the same notation as in (a). Since this time the calculation of \( \mathbb{P}[B] \) is considerably more involved, we again look at \( \mathbb{P}[A] \) and \( \mathbb{P}[C] \). By assumption, conditionally given \( \Theta, N_1, \ldots, N_6 \) are independent Poisson random variables with frequency parameter \( \Theta \lambda \), where \( \lambda = 0.2 \). Therefore, we can calculate

\[
\mathbb{P}[A] = \mathbb{P}[N_1 = 0, \ldots, N_6 = 0] = \mathbb{E}[\mathbb{P}[N_1 = 0, \ldots, N_6 = 0|\Theta]]
\]

\[
= \mathbb{E} \left[ \prod_{i=1}^{6} \mathbb{P}[N_i = 0|\Theta] \right]
\]

\[
= \mathbb{E} \left[ \prod_{i=1}^{6} \exp\left\{-\Theta \lambda \right\} \right]
\]

\[
= \mathbb{E}[\exp\left\{-6\Theta \lambda \right\}]
\]

\[
= M_\Theta(-6\lambda),
\]

where \( M_\Theta \) denotes the moment generating function of \( \Theta \). Since \( \Theta \sim \Gamma(1, 1) \), \( M_\Theta \) is given by

\[
M_\Theta(r) = \frac{1}{1 - r},
\]

for all \( r < 1 \), which leads to

\[
\mathbb{P}[A] = \frac{1}{1 + 6\lambda} = \frac{1}{2.2}.
\]

Similarly, we get

\[
\mathbb{P}[C] = 1 - \mathbb{P}[C^c] = 1 - \mathbb{P}[N_1 = 0, N_5 = 0, N_6 = 0] = 1 - \frac{1}{1 + 3\lambda} = 1 - \frac{1}{1.6} = \frac{0.6}{1.6}.
\]

For the event \( B \) we get

\[
\mathbb{P}[B] = 1 - \mathbb{P}[A] - \mathbb{P}[C] = 1 - \frac{1}{2.2} - \frac{0.6}{1.6} = \frac{1}{1.6} - \frac{1}{2.2}.
\]

Thus the expected proportion \( q \) of the base premium that is still paid after the grant of the no-claims bonus is given by

\[
q = 0.8 \cdot \mathbb{P}[A] + 0.9 \cdot \mathbb{P}[B] + 1 \cdot \mathbb{P}[C]
\]

\[
= 0.8 \cdot \frac{1}{2.2} + 0.9 \cdot \left( \frac{1}{1.6} - \frac{1}{2.2} \right) + \frac{0.6}{1.6}
\]

\[
\approx 0.892.
\]

We conclude that the surcharge \( s \) on the base premium is given by

\[
s = \frac{1}{q} - 1 \approx 12.1%,
\]

which is considerably bigger than in (a).

**Solution 3.2 Central Limit Theorem**

Let \( \sigma^2 \) be the variance of the claim sizes and \( x > 0 \). Then we have

\[
\mathbb{P}\left[ \left| \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu \right| < \frac{x}{\sqrt{n}} \right] = \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu < \frac{x}{\sqrt{n}} \right] - \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu > \frac{x}{\sqrt{n}} \right]
\]

\[
= \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu < \frac{x}{\sqrt{n}} \right] - \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu > \frac{x}{\sqrt{n}} \right]
\]

\[
= \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu < \frac{x}{\sqrt{n}} \right] - \mathbb{P}\left[ \left| \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu \right| > \frac{x}{\sqrt{n}} \right]
\]

\[
= \mathbb{P}\left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_i - \mu}{\sigma} < \frac{x}{\sigma} \right] - \mathbb{P}\left[ \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_i - \mu}{\sigma} \right| > \frac{x}{\sigma} \right],
\]

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where
\[ Z_n = \sqrt{n} \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu. \]

According to the Central Limit Theorem, \( Z_n \) converges in distribution to a standard Gaussian random variable. Hence, if we write \( \Phi \) for the distribution function of a standard Gaussian random variable, we have the approximation
\[
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_i - \mu \right| < \frac{x}{\sqrt{n}} \right) \approx \Phi \left( \frac{x}{\sigma} \right) - \Phi \left( -\frac{x}{\sigma} \right).
\]

On the one hand, as we are interested in a probability of at least 95%, we have to choose \( x > 0 \) such that
\[
\Phi \left( \frac{x}{\sigma} \right) - \Phi \left( -\frac{x}{\sigma} \right) = 0.95,
\]
which implies
\[
x = 1.96 \cdot \sigma = 1.96 \cdot \text{Vco}(Y_1) \cdot \mu = 1.96 \cdot 4 \cdot \mu. \tag{1}
\]

On the other hand, as we want the deviation of the empirical mean from \( \mu \) to be less than 1%, we set
\[
x \sqrt{n} = 0.01 \cdot \mu,
\]
which implies
\[
n = \frac{x^2}{0.01^2 \cdot \mu^2}. \tag{2}
\]

Combining (1) and (2), we conclude
\[
n = \frac{(1.96 \cdot 4 \cdot \mu)^2}{0.01^2 \cdot \mu^2} = 1.96^2 \cdot 4^2 \cdot 10'000 = 614'656.
\]

**Solution 3.3 Compound Binomial Distribution**

For \( \tilde{S} \sim \text{CompBinom}(\tilde{v}, \tilde{p}, \tilde{G}) \) with the random variable \( \tilde{Y}_1 \) having distribution function \( \tilde{G} \) and moment generating function \( M_{\tilde{Y}_1} \), the moment generating function \( M_{\tilde{S}} \) of \( \tilde{S} \) is given by
\[
M_{\tilde{S}}(r) = (\tilde{p} M_{\tilde{Y}_1}(r) + 1 - \tilde{p})^{\tilde{v}},
\]
for all \( r \in \mathbb{R} \) for which \( M_{\tilde{Y}_1} \) is defined. We calculate the moment generating function of \( S_{lc} \) and show that it is exactly of the form given above. Since \( S_{lc} \geq 0 \) almost surely, its moment generating function is defined at least for all \( r < 0 \). Thus, for \( r < 0 \), we have
\[
M_{S_{lc}}(r) = \mathbb{E} \left[ \exp \left\{ r \sum_{i=1}^{N} Y_i 1_{\{Y_i > M\}} \right\} \right]
\]
\[
= \mathbb{E} \left[ \prod_{i=1}^{N} \exp \left\{ r Y_i 1_{\{Y_i > M\}} \right\} \right]
\]
\[
= \mathbb{E} \left[ \prod_{i=1}^{N} \mathbb{E} \left[ \exp \left\{ r Y_i 1_{\{Y_i > M\}} \right\} \mid N \right] \right]
\]
\[
= \mathbb{E} \left[ \prod_{i=1}^{N} \mathbb{E} \left[ \exp \left\{ r Y_i 1_{\{Y_i > M\}} \right\} \right] \right].
\]
where in the third equality we used the tower property of conditional expectation and in the fourth equality the independence between $N$ and $Y_i$. For the inner expectation we get

$$
E \left[ \exp \left\{ r Y_i \ 1_{\{Y_i > M\}} \right\} \right] = E \left[ \exp \left\{ r Y_i \right\} \left| Y_i > M \right\} \mathbb{P}[Y_i > M] + \mathbb{P}[Y_i \leq M] 
= E \left[ \exp \left\{ r Y_i \right\} \left| Y_i > M \right\} \left[ 1 - G(M) \right] + G(M) \right].
$$

First note that the distribution function of the random variable $Y_i \mid Y_i > M$ is $G_{lc}$. Moreover, since $Y_i \mid Y_i > M$ is greater than 0 almost surely, its moment generating function $M_{Y_i \mid Y_i > M}$ is defined for all $r < 0$ and thus we can write

$$
E \left[ \exp \left\{ r Y_i \ 1_{\{Y_i > M\}} \right\} \right] = M_{Y_i \mid Y_i > M}(r) \left[ 1 - G(M) \right] + G(M).
$$

Hence we get

$$
M_{S_{lc}}(r) = E \left[ \prod_{i=1}^{N} (M_{Y_i \mid Y_i > M}(r) \left[ 1 - G(M) \right] + G(M)) \right]
= E \left[ \left( M_{Y_i \mid Y_i > M}(r) \left[ 1 - G(M) \right] + G(M) \right) ^ N \right]
= E \left[ \exp \left\{ N \log (M_{Y_i \mid Y_i > M}(r) \left[ 1 - G(M) \right] + G(M)) \right\} \right]
= M_{N}(\rho),
$$

where $M_{N}$ is the moment generating function of $N$ and

$$
\rho = \log (M_{Y_i \mid Y_i > M}(r) \left[ 1 - G(M) \right] + G(M)).
$$

Since we have $N \sim \text{Binom}(v, p)$, $M_{N}(r)$ is given by

$$
M_{N}(r) = (p \exp \{r\} + 1 - p)^v.
$$

Therefore, we get

$$
M_{S_{lc}}(r) = [p \left( M_{Y_i \mid Y_i > M}(r) \left[ 1 - G(M) \right] + G(M) \right) + 1 - p]^v
= (p[1 - G(M)]M_{Y_i \mid Y_i > M}(r) + 1 - p[1 - G(M)])^v.
$$

Applying Lemma 1.3 of the lecture notes, we conclude that $S_{lc} \sim \text{CompBinom}(\tilde{v}, \tilde{p}, \tilde{G})$ with $\tilde{v} = v, \tilde{p} = p[1 - G(M)]$ and $\tilde{G} = G_{lc}$.