Non-Life Insurance: Mathematics and Statistics

Solution sheet 3

Solution 3.1 No-Claims Bonus

(a) We define the following events:

$$A = \{\text{"no claims in the last six years"}\},$$
$$B = \{\text{"no claims in the last three years but at least one claim in the last six years"}\},$$
$$C = \{\text{"at least one claim in the last three years"}\}.$$

Note that since the events A, B and C are disjoint and cover all possible outcomes, we have

$$\mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] = 1,$$

i.e. it is sufficient to calculate two out of the three probabilities. Since the calculation of $\mathbb{P}[B]$ is slightly more involved, we will look at $\mathbb{P}[A]$ and $\mathbb{P}[C]$. Let $N_1, \ldots, N_6$ be the number of claims of the last six years of our considered car driver, where N_6 corresponds to the most recent year. By assumption, $N_1, \ldots, N_6$ are independent Poisson random variables with frequency parameter $\lambda = 0.2$. Therefore, we can calculate

$$\mathbb{P}[A] = \mathbb{P}\left[N_1 = 0, \ldots, N_6 = 0\right] = \prod_{i=1}^{6} \mathbb{P}\left[N_i = 0\right] = \prod_{i=1}^{6} \exp\{-\lambda\} = \exp\{-6\lambda\} = \exp\{-1.2\}$$

and, similarly,

$$\mathbb{P}[C] = 1 - \mathbb{P}[C^c] = 1 - \mathbb{P}\left[N_4 = 0, N_5 = 0, N_6 = 0\right] = 1 - \exp\{-3\lambda\} = 1 - \exp\{-0.6\}.$$

For the event B we get

$$\mathbb{P}[B] = 1 - \mathbb{P}[A] - \mathbb{P}[C] = 1 - \exp\{-1.2\} - (1 - \exp\{-0.6\}) = \exp\{-0.6\} - \exp\{-1.2\}.$$

Thus the expected proportion q of the base premium that is still paid after the grant of the no-claims bonus is given by

$$\begin{aligned} q &= 0.8 \cdot \mathbb{P}[A] + 0.9 \cdot \mathbb{P}[B] + 1 \cdot \mathbb{P}[C] \\ &= 0.8 \cdot \exp\{-1.2\} + 0.9 \cdot (\exp\{-0.6\} - \exp\{-1.2\}) + 1 - \exp\{-0.6\} \\ &\approx 0.915. \end{aligned}$$

If s denotes the surcharge on the base premium, then it has to satisfy the equation

$$q(1 + s) \cdot \text{base premium} = \text{base premium},$$

which leads to

$$s = \frac{1}{q} - 1.$$

We conclude that the surcharge on the base premium is given by approximately 9.3%.

(b) We use the same notation as in (a). Since this time the calculation of $\mathbb{P}[B]$ is considerably more involved, we again look at $\mathbb{P}[A]$ and $\mathbb{P}[C]$. By assumption, conditionally given Θ, $N_1, \dots, N_6$ are independent Poisson random variables with frequency parameter $\Theta\lambda$, where $\lambda = 0.2$. Therefore, we can calculate

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}\left[N_1 = 0, \dots, N_6 = 0\right] \\ &= \mathbb{E}\left[\mathbb{P}\left[N_1 = 0, \dots, N_6 = 0 | \Theta\right]\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{6} \mathbb{P}\left[N_i = 0 | \Theta\right]\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{6} \exp\{-\Theta\lambda\}\right] \\ &= \mathbb{E}\left[\exp\{-6\Theta\lambda\}\right] \\ &= M_\Theta(-6\lambda),\end{aligned}$$

where M_Θ denotes the moment generating function of Θ. Since $\Theta \sim \Gamma(1, 1)$, M_Θ is given by

$$M_\Theta(r) = \frac{1}{1-r},$$

for all $r < 1$, which leads to

$$\mathbb{P}[A] = \frac{1}{1+6\lambda} = \frac{1}{2.2}.$$

Similarly, we get

$$\mathbb{P}[C] = 1 - \mathbb{P}[C^c] = 1 - \mathbb{P}\left[N_4 = 0, N_5 = 0, N_6 = 0\right] = 1 - \frac{1}{1+3\lambda} = 1 - \frac{1}{1.6} = \frac{0.6}{1.6}.$$

For the event B we get

$$\mathbb{P}[B] = 1 - \mathbb{P}[A] - \mathbb{P}[C] = 1 - \frac{1}{2.2} - \frac{0.6}{1.6} = \frac{1}{1.6} - \frac{1}{2.2}.$$

Thus the expected proportion q of the base premium that is still paid after the grant of the no-claims bonus is given by

$$\begin{aligned}q &= 0.8 \cdot \mathbb{P}[A] + 0.9 \cdot \mathbb{P}[B] + 1 \cdot \mathbb{P}[C] \\ &= 0.8 \cdot \frac{1}{2.2} + 0.9 \cdot \left(\frac{1}{1.6} - \frac{1}{2.2}\right) + \frac{0.6}{1.6} \\ &\approx 0.892.\end{aligned}$$

We conclude that the surcharge s on the base premium is given by

$$s = \frac{1}{q} - 1 \approx 12.1\%,$$

which is considerably bigger than in (a).

Solution 3.2 Central Limit Theorem

Let σ^2 be the variance of the claim sizes and $x > 0$. Then we have

$$\begin{aligned}\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} Y_i - \mu\right| < \frac{x}{\sqrt{n}}\right] &= \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i - \mu < \frac{x}{\sqrt{n}}\right] - \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i - \mu \le -\frac{x}{\sqrt{n}}\right] \\ &= \mathbb{P}\left[\sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^{n} Y_i - \mu}{\sigma} < \frac{x}{\sigma}\right] - \mathbb{P}\left[\sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^{n} Y_i - \mu}{\sigma} \le -\frac{x}{\sigma}\right] \\ &= \mathbb{P}\left[Z_n < \frac{x}{\sigma}\right] - \mathbb{P}\left[Z_n \le -\frac{x}{\sigma}\right],\end{aligned}$$

where

$$Z_n = \sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^n Y_i - \mu}{\sigma}.$$

According to the Central Limit Theorem, Z_n converges in distribution to a standard Gaussian random variable. Hence, if we write Φ for the distribution function of a standard Gaussian random variable, we have the approximation

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^n Y_i - \mu\right| < \frac{x}{\sqrt{n}}\right] \approx \Phi\left(\frac{x}{\sigma}\right) - \Phi\left(-\frac{x}{\sigma}\right).$$

On the one hand, as we are interested in a probabilty of at least 95%, we have to choose $x > 0$ such that

$$\Phi\left(\frac{x}{\sigma}\right) - \Phi\left(-\frac{x}{\sigma}\right) = 0.95,$$

which implies

$$\frac{x}{\sigma} = 1.96.$$

It follows that

$$x = 1.96 \cdot \sigma = 1.96 \cdot \mathrm{Vco}(Y_1) \cdot \mu = 1.96 \cdot 4 \cdot \mu. \tag{1}$$

On the other hand, as we want the deviation of the empirical mean from μ to be less than 1%, we set

$$\frac{x}{\sqrt{n}} = 0.01 \cdot \mu,$$

which implies

$$n = \frac{x^2}{0.01^2 \cdot \mu^2}. \tag{2}$$

Combining (1) and (2), we conclude

$$n = \frac{(1.96 \cdot 4 \cdot \mu)^2}{0.01^2 \cdot \mu^2} = 1.96^2 \cdot 4^2 \cdot 10'000 = 614'656.$$

Solution 3.3 Compound Binomial Distribution

For $\tilde{S} \sim \mathrm{CompBinom}(\tilde{v}, \tilde{p}, \tilde{G})$ with the random variable $\tilde{Y}_1$ having distribution function $\tilde{G}$ and moment generating function $M_{\tilde{Y}_1}$, the moment generating function $M_{\tilde{S}}$ of $\tilde{S}$ is given by

$$M_{\tilde{S}}(r) = \left(\tilde{p}M_{\tilde{Y}_1}(r) + 1 - \tilde{p}\right)^{\tilde{v}},$$

for all $r \in \mathbb{R}$ for which $M_{\tilde{Y}_1}$ is defined. We calculate the moment generating function of S_{lc} and show that it is exactly of the form given above. Since $S_{\mathrm{lc}} \geq 0$ almost surely, its moment generating function is defined at least for all $r < 0$. Thus, for $r < 0$, we have

$$\begin{aligned}
M_{S_{\mathrm{lc}}}(r) &= \mathbb{E}\left[\exp\left\{r\sum_{i=1}^N Y_i \ 1_{\{Y_i > M\}}\right\}\right] \\
&= \mathbb{E}\left[\prod_{i=1}^N \exp\left\{rY_i \ 1_{\{Y_i > M\}}\right\}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^N \exp\left\{rY_i \ 1_{\{Y_i > M\}}\right\}\middle| N\right]\right] \\
&= \mathbb{E}\left[\prod_{i=1}^N \mathbb{E}\left[\exp\left\{rY_i \ 1_{\{Y_i > M\}}\right\}\right]\right],
\end{aligned}$$

where in the third equality we used the tower property of conditional expectation and in the fourth equality the independence between N and Y_i. For the inner expectation we get

$$\begin{aligned}\mathbb{E}\left[\exp\left\{rY_i\ 1_{\{Y_i>M\}}\right\}\right] &= \mathbb{E}\left[\exp\{rY_i\}\cdot 1_{\{Y_i>M\}} + 1_{\{Y_i\le M\}}\right]\\ &= \mathbb{E}\left[\exp\{rY_i\}\,|Y_i>M\right]\mathbb{P}[Y_i>M] + \mathbb{P}[Y_i\le M]\\ &= \mathbb{E}\left[\exp\{rY_i\}\,|Y_i>M\right][1-G(M)] + G(M).\end{aligned}$$

First note that the distribution function of the random variable $Y_i \mid Y_i > M$ is G_{lc}. Moreover, since $Y_i \mid Y_i > M$ is greater than 0 almost surely, its moment generating function $M_{Y_1|Y_1>M}$ is defined for all $r<0$ and thus we can write

$$\mathbb{E}\left[\exp\left\{rY_i\ 1_{\{Y_i>M\}}\right\}\right] = M_{Y_1|Y_1>M}(r)[1-G(M)] + G(M).$$

Hence we get

$$\begin{aligned}M_{S_{\text{lc}}}(r) &= \mathbb{E}\left[\prod_{i=1}^{N}\left(M_{Y_1|Y_1>M}(r)[1-G(M)]+G(M)\right)\right]\\ &= \mathbb{E}\left[\left(M_{Y_1|Y_1>M}(r)[1-G(M)]+G(M)\right)^N\right]\\ &= \mathbb{E}\left[\exp\left\{N\log\left(M_{Y_1|Y_1>M}(r)[1-G(M)]+G(M)\right)\right\}\right]\\ &= M_N(\rho),\end{aligned}$$

where M_N is the moment generating function of N and

$$\rho = \log\left(M_{Y_1|Y_1>M}(r)[1-G(M)]+G(M)\right).$$

Since we have $N\sim \text{Binom}(v,p)$, $M_N(r)$ is given by

$$M_N(r) = (p\exp\{r\}+1-p)^v.$$

Therefore, we get

$$\begin{aligned}M_{S_{\text{lc}}}(r) &= \left[p\left(M_{Y_1|Y_1>M}(r)[1-G(M)]+G(M)\right)+1-p\right]^v\\ &= \left(p[1-G(M)]M_{Y_1|Y_1>M}(r)+1-p[1-G(M)]\right)^v.\end{aligned}$$

Applying Lemma 1.3 of the lecture notes, we conclude that $S_{\text{lc}}\sim \text{CompBinom}(\tilde{v},\tilde{p},\tilde{G})$ with $\tilde{v}=v$, $\tilde{p}=p[1-G(M)]$ and $\tilde{G}=G_{\text{lc}}$.