Non-Life Insurance: Mathematics and Statistics

Solution sheet 4

Solution 4.1 Poisson Model and Negative-Binomial Model

(a) Let $v = v_1 = \cdots = v_{10} = 10'000$. In the Poisson model we assume that $N_1, \ldots, N_{10}$ are independent with $N_t \sim \text{Poi}(\lambda v_t)$ for all $t \in \{1, \ldots, 10\}$. We use Estimator 2.32 of the lecture notes to estimate the claims frequency parameter λ by

$$\hat{\lambda}_{10}^{\text{MLE}} = \frac{\sum_{t=1}^{10} N_t}{\sum_{t=1}^{10} v_t} = \frac{\sum_{t=1}^{10} N_t}{10v} = \frac{10'224}{100'000} \approx 10.22\%.$$

Note that a random variable $N \sim \text{Poi}(\lambda v)$ can be understood as

$$N \overset{(\text{d})}{=} \sum_{i=1}^{v} N_i, \tag{1}$$

where $N_1, \ldots, N_v$ are independent random variables that all follow a $\text{Poi}(\lambda)$-distribution. If we define $\hat{\lambda} = N/v$, then we have

$$\mathbb{E}\left[\hat{\lambda}\right] = \mathbb{E}\left[\frac{N}{v}\right] = \frac{\mathbb{E}[N]}{v} = \frac{\lambda v}{v} = \lambda,$$

hence $\hat{\lambda}$ can be seen as an estimator for λ. Moreover, we have

$$\text{Var}\left(\hat{\lambda}\right) = \text{Var}\left(\frac{N}{v}\right) = \frac{\text{Var}(N)}{v^2} = \frac{\lambda v}{v^2} = \frac{\lambda}{v}$$

and, because of (1), we can use the Central Limit Theorem to get

$$\frac{N/v - \mathbb{E}\left[N/v\right]}{\sqrt{\text{Var}\left(N/v\right)}} = \frac{\hat{\lambda} - \lambda}{\sqrt{\lambda/v}} \longrightarrow Z,$$

as $v \to \infty$, where Z is a random variable following a standard normal distribution. Hence, we have the approximation

$$\mathbb{P}\left[\hat{\lambda} - \sqrt{\frac{\lambda}{v}} \leq \lambda \leq \hat{\lambda} + \sqrt{\frac{\lambda}{v}}\right] = \mathbb{P}\left[-1 \leq \frac{\hat{\lambda} - \lambda}{\sqrt{\lambda/v}} \leq 1\right] \approx \mathbb{P}(-1 \leq Z \leq 1) \approx 0.7,$$

i.e. with a probability of roughly 70%, λ lies in the interval $\left[\hat{\lambda} - \sqrt{\lambda/v}, \hat{\lambda} + \sqrt{\lambda/v}\right]$. Since a confidence interval for λ is not allowed to depend on λ itself, we also replace it by the estimator $\hat{\lambda}$ to get an approximate, roughly 70%-confidence interval $\left[\hat{\lambda} - \sqrt{\hat{\lambda}/v}, \hat{\lambda} + \sqrt{\hat{\lambda}/v}\right]$ for λ. If we look at the estimator $\hat{\lambda}_{10}^{\text{MLE}}$ as the random variable $\left(\sum_{t=1}^{10} N_t\right) / (10v)$, we see that

$$\mathbb{E}\left[\hat{\lambda}_{10}^{\text{MLE}}\right] = \frac{\sum_{t=1}^{10} \mathbb{E}[N_t]}{10v} = \frac{\sum_{t=1}^{10} \lambda v_t}{10v} = \lambda = \mathbb{E}\left[\hat{\lambda}\right]$$

and

$$\text{Var}\left(\hat{\lambda}_{10}^{\text{MLE}}\right) = \frac{\sum_{t=1}^{10} \text{Var}(N_t)}{(10v)^2} = \frac{\sum_{t=1}^{10} \lambda v_t}{(10v)^2} = \frac{\lambda}{10v} < \frac{\lambda}{v} = \text{Var}\left(\hat{\lambda}\right).$$

Because of the smaller variance it makes sense to replace $\hat{\lambda}$ by $\hat{\lambda}_{10}^{\mathrm{MLE}}$ to get the approximate, roughly 70%-confidence interval

$$\left[\hat{\lambda}_{10}^{\mathrm{MLE}} - \sqrt{\frac{\hat{\lambda}_{10}^{\mathrm{MLE}}}{v}}, \hat{\lambda}_{10}^{\mathrm{MLE}} + \sqrt{\frac{\hat{\lambda}_{10}^{\mathrm{MLE}}}{v}}\right] \approx [9.90\%, 10.54\%]$$

for λ. If we define $\lambda_t = N_t/v_t$ for all $t \in \{1, \dots, 10\}$, we have the following observations $\lambda_1, \dots, \lambda_{10}$ of the frequency parameter λ:

t	1	2	3	4	5	6	7	8	9	10
$\lambda_t = \frac{N_t}{v_t}$	10%	9.97%	9.85%	9.89%	10.56%	10.70%	9.94%	9.86%	10.93%	10.54%

Table 1: Observed claims frequencies $\lambda_t = N_t/v_t$.

We observe that instead of the expected, roughly seven observations, only four observations lie in the estimated confidence interval. We conclude that the assumption of having Poisson distributions might not be reasonable.

(b) By equation (2.8) of the lecture notes, the test statistic $\hat{\chi}^*$ is given by

$$\hat{\chi}^* = \sum_{t=1}^{10} v_t \frac{\left(N_t/v_t - \hat{\lambda}_{10}^{\mathrm{MLE}}\right)^2}{\hat{\lambda}_{10}^{\mathrm{MLE}}}$$

and is approximately χ^2-distributed with $10 - 1 = 9$ degrees of freedom. By inserting the numbers and $\hat{\lambda}_{10}^{\mathrm{MLE}}$ calculated in (a), we get

$$\hat{\chi}^* \approx 14.84.$$

The probability that a random variable with a χ^2-distribution with 9 degrees of freedom is greater than 14.84 is approximately equal to 9.55%. Hence we can reject the null hypothesis of having Poisson distributions only at significance levels that are higher than 9.55%. In particular, we can not reject the null hypothesis at the significance level of 5%.

(c) As in part (a), let $v = v_1 = \dots = v_{10} = 10\text{'}000$. In the negative-binomial model we assume that $N_1, \dots, N_{10}$ are independent with $N_t \sim \mathrm{Poi}(\Theta_t \lambda v_t)$ for all $t \in \{1, \dots, 10\}$, where $\Theta_1, \dots, \Theta_{10} \overset{\text{i.i.d.}}{\sim} \Gamma(\gamma, \gamma)$ for some $\gamma > 0$. We use Estimator 2.28 of the lecture notes to estimate the claims frequency parameter λ by

$$\hat{\lambda}_{10}^{\mathrm{NB}} = \frac{\sum_{t=1}^{10} N_t}{\sum_{t=1}^{10} v_t} = \frac{\sum_{t=1}^{10} N_t}{10v} = \frac{10\text{'}224}{100\text{'}000} \approx 10.22\%.$$

As in equation (2.7) of the lecture notes, we define

$$\hat{V}_{10}^2 = \frac{1}{9} \sum_{t=1}^{10} v_t \left(\frac{N_t}{v_t} - \hat{\lambda}_{10}^{\mathrm{NB}}\right) \approx 16.9\%.$$

Now we can use Estimator 2.30 of the lecture notes to estimate the dispersion parameter γ by

$$\hat{\gamma}_{10}^{\mathrm{NB}} = \frac{\left(\hat{\lambda}_{10}^{\mathrm{NB}}\right)^2}{\hat{V}_{10}^2 - \hat{\lambda}_{10}^{\mathrm{NB}}} \frac{1}{9} \left(\sum_{t=1}^{10} v_t - \frac{\sum_{t=1}^{10} v_t^2}{\sum_{t=1}^{10} v_t}\right) = \frac{\left(\hat{\lambda}_{10}^{\mathrm{NB}}\right)^2}{\hat{V}_{10}^2 - \hat{\lambda}_{10}^{\mathrm{NB}}} \frac{\left(10v - \frac{10v^2}{10v}\right)}{9} = \frac{\left(\hat{\lambda}_{10}^{\mathrm{NB}}\right)^2 v}{\hat{V}_{10}^2 - \hat{\lambda}_{10}^{\mathrm{NB}}} \approx 1576.15.$$

For a random variable $N \sim \text{Poi}(\Theta\lambda v)$, conditionally given Θ, we have

$$\mathbb{E}\left[\frac{N}{v}\right] = \frac{\mathbb{E}[N]}{v} = \frac{\mathbb{E}[\mathbb{E}[N|\Theta]]}{v} = \frac{\mathbb{E}[\Theta\lambda v]}{v} = \frac{\lambda v}{v} = \lambda,$$

since $\mathbb{E}[\Theta] = 1$, and

$$\text{Var}\left(\frac{N}{v}\right) = \frac{\mathbb{E}[\text{Var}(N|\Theta)] + \text{Var}(\mathbb{E}[N|\Theta])}{v^2} = \frac{\mathbb{E}[\Theta\lambda v] + \text{Var}(\Theta\lambda v)}{v^2} = \frac{\lambda v + \frac{\lambda^2 v^2}{\gamma}}{v^2} = \frac{\lambda + \frac{\lambda^2 v}{\gamma}}{v},$$

since $\text{Var}(\Theta) = 1/\gamma$. Similarly as in the Poisson case in part (a), we get the approximate, roughly 70%-confidence interval

$$\left[\hat{\lambda}_{10}^{\text{NB}} - \sqrt{\frac{\hat{\lambda}_{10}^{\text{NB}} + \left(\hat{\lambda}_{10}^{\text{NB}}\right)^2 v/\hat{\gamma}_{10}^{\text{NB}}}{v}}, \hat{\lambda}_{10}^{\text{NB}} + \sqrt{\frac{\hat{\lambda}_{10}^{\text{NB}} + \left(\hat{\lambda}_{10}^{\text{NB}}\right)^2 v/\hat{\gamma}_{10}^{\text{NB}}}{v}}\right] \approx [9.81\%, 10.63\%].$$

for λ. Looking at the observations $\lambda_1, \ldots, \lambda_{10}$ given in Table 1 above, we see that eight of them lie in the estimated confidence interval, which is clearly better than in the Poisson case in part (a). In conclusion, the negative-binomial model seems more reasonable than the Poisson model.

Solution 4.2 Compound Poisson Distribution

(a) Since $S \sim \text{CompPoi}(\lambda v, G)$, we can write S as

$$S = \sum_{i=1}^{N} Y_i,$$

where $N \sim \text{Poi}(\lambda v)$, $Y_1, Y_2, \ldots$ are i.i.d. with distribution function G and N and $Y_1, Y_2, \ldots$ are independent. Now we can define S_{sc}, S_{mc} and S_{lc} as

$$S_{\text{sc}} = \sum_{i=1}^{N} Y_i 1_{\{Y_i \leq 1'000\}}, \quad S_{\text{mc}} = \sum_{i=1}^{N} Y_i 1_{\{1'000 < Y_i \leq 1'000'000\}} \quad \text{and} \quad S_{\text{lc}} = \sum_{i=1}^{N} Y_i 1_{\{Y_i > 1'000'000\}}.$$

(b) Note that according to Table 2 given on the exercise sheet, we have

$$\begin{aligned}
\mathbb{P}[Y_1 \leq 1'000] &= \mathbb{P}[Y = 100] + \mathbb{P}[Y = 300] + \mathbb{P}[Y = 500] = \frac{3}{20} + \frac{4}{20} + \frac{3}{20} = \frac{1}{2}, \\
\mathbb{P}[1'000 < Y_1 \leq 1'000'000] &= \mathbb{P}[Y = 6'000] + \mathbb{P}[Y = 100'000] + \mathbb{P}[Y = 500'000] \\
&= \frac{2}{15} + \frac{2}{15} + \frac{1}{15} \\
&= \frac{1}{3} \quad \text{and} \\
\mathbb{P}[Y_1 > 1'000'000] &= 1 - \mathbb{P}[Y_1 \leq 1'000'000] = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
\end{aligned}$$

Thus, using Theorem 2.14 of the lecture notes (disjoint decomposition of compound Poisson distributions), we get

$$S_{\text{sc}} \sim \text{CompPoi}\left(\frac{\lambda v}{2}, G_{\text{sc}}\right), \quad S_{\text{mc}} \sim \text{CompPoi}\left(\frac{\lambda v}{3}, G_{\text{mc}}\right) \quad \text{and} \quad S_{\text{lc}} \sim \text{CompPoi}\left(\frac{\lambda v}{6}, G_{\text{lc}}\right),$$

where

$$G_{\text{sc}}(y) = \mathbb{P}[Y_1 \le y | Y_1 \le 1\text{'}000],$$
$$G_{\text{mc}}(y) = \mathbb{P}[Y_1 \le y | 1\text{'}000 < Y_1 \le 1\text{'}000\text{'}000] \quad \text{and}$$
$$G_{\text{lc}}(y) = \mathbb{P}[Y_1 \le y | Y_1 > 1\text{'}000\text{'}000]$$

for all $y \in \mathbb{R}$. In particular, for a random variable Y_{sc} having distribution function G_{sc}, we have

$$\mathbb{P}[Y_{\text{sc}} = 100] = \frac{\mathbb{P}[Y = 100]}{\mathbb{P}[Y_1 \le 1\text{'}000]} = \frac{3/20}{1/2} = \frac{3}{10},$$
$$\mathbb{P}[Y_{\text{sc}} = 300] = \frac{\mathbb{P}[Y = 300]}{\mathbb{P}[Y_1 \le 1\text{'}000]} = \frac{4/20}{1/2} = \frac{4}{10} \quad \text{and}$$
$$\mathbb{P}[Y_{\text{sc}} = 500] = \frac{\mathbb{P}[Y = 500]}{\mathbb{P}[Y_1 \le 1\text{'}000]} = \frac{3/20}{1/2} = \frac{3}{10}$$

Analogously, for random variables Y_{mc} and Y_{lc} having distribution functions G_{mc} and G_{lc}, respectively, we get

$$\mathbb{P}[Y_{\text{mc}} = 6\text{'}000] = \frac{2}{5}, \quad \mathbb{P}[Y_{\text{mc}} = 100\text{'}000] = \frac{2}{5} \quad \text{and} \quad \mathbb{P}[Y_{\text{mc}} = 500\text{'}000] = \frac{1}{5},$$

as well as

$$\mathbb{P}[Y_{\text{lc}} = 2\text{'}000\text{'}000] = \frac{1}{2}, \quad \mathbb{P}[Y_{\text{lc}} = 5\text{'}000\text{'}000] = \frac{1}{4} \quad \text{and} \quad \mathbb{P}[Y_{\text{lc}} = 10\text{'}000\text{'}000] = \frac{1}{4}.$$

(c) According to Theorem 2.14 of the lecture notes, S_{sc}, S_{mc} and S_{lc} are independent.

(d) In order to find $\mathbb{E}[S_{\text{sc}}]$, we need $\mathbb{E}[Y_{\text{sc}}]$, which can be calculated as

$$\mathbb{E}[Y_{\text{sc}}] = 100 \cdot \mathbb{P}[Y_{\text{sc}} = 100] + 300 \cdot \mathbb{P}[Y_{\text{mc}} = 300] + 500 \cdot \mathbb{P}[Y_{\text{lc}} = 500] = \frac{300}{10} + \frac{1200}{10} + \frac{1500}{10} = 300.$$

Now we can apply Proposition 2.11 of the lecture notes to get

$$\mathbb{E}[S_{\text{sc}}] = \frac{\lambda v}{2} \mathbb{E}[Y_{\text{sc}}] = 0.3 \cdot 300 = 90.$$

Similarly, we get

$$\mathbb{E}[Y_{\text{mc}}] = 142\text{'}400 \quad \text{and} \quad \mathbb{E}[Y_{\text{lc}}] = 4\text{'}750\text{'}000.$$

Thus we find

$$\mathbb{E}[S_{\text{mc}}] = \frac{\lambda v}{3} \mathbb{E}[Y_{\text{mc}}] = 28\text{'}480 \quad \text{and} \quad \mathbb{E}[S_{\text{lc}}] = \frac{\lambda v}{6} \mathbb{E}[Y_{\text{lc}}] = 475\text{'}000.$$

Since $S = S_{\text{sc}} + S_{\text{mc}} + S_{\text{lc}}$, we get

$$\mathbb{E}[S] = \mathbb{E}[S_{\text{sc}}] + \mathbb{E}[S_{\text{mc}}] + \mathbb{E}[S_{\text{lc}}] = 503\text{'}570.$$

In order to find $\text{Var}(S_{\text{sc}})$, we need $\mathbb{E}[Y_{\text{sc}}^2]$, which can be calculated as

$$\mathbb{E}[Y_{\text{sc}}^2] = 100^2 \cdot \mathbb{P}[Y_{\text{sc}} = 100] + 300^2 \cdot \mathbb{P}[Y_{\text{mc}} = 300] + 500^2 \cdot \mathbb{P}[Y_{\text{lc}} = 500]$$
$$= \frac{30\text{'}000}{10} + \frac{360\text{'}000}{10} + \frac{750\text{'}000}{10} = 114\text{'}000.$$

Now we can apply Proposition 2.11 of the lecture notes to get

$$\text{Var}(S_{\text{sc}}) = \frac{\lambda v}{2} \mathbb{E}[Y_{\text{sc}}^2] = 0.3 \cdot 114\text{'}000 = 34\text{'}200.$$

Similarly, we get

$$\mathbb{E}[Y_{\mathrm{mc}}^2] = 54'014'400'000 \quad \text{and} \quad \mathbb{E}[Y_{\mathrm{lc}}^2] = 33'250'000'000'000.$$

Thus we find

$$\mathrm{Var}(S_{\mathrm{mc}}) = \frac{\lambda v}{3}\mathbb{E}[Y_{\mathrm{mc}}^2] = 10'802'880'000 \quad \text{and} \quad \mathrm{Var}(S_{\mathrm{lc}}) = \frac{\lambda v}{6}\mathbb{E}[Y_{\mathrm{lc}}^2] = 3'325'000'000'000.$$

Since $S = S_{\mathrm{sc}} + S_{\mathrm{mc}} + S_{\mathrm{lc}}$ and S_{sc}, S_{mc} and S_{lc} are independent, we get

$$\mathrm{Var}(S) = \mathrm{Var}(S_{\mathrm{sc}}) + \mathrm{Var}(S_{\mathrm{mc}}) + \mathrm{Var}(S_{\mathrm{lc}}) = 3'335'802'914'200.$$

(e) First, we define the random variable N_{lc} as

$$N_{\mathrm{lc}} \sim \mathrm{Poi}\left(\frac{\lambda v}{6}\right).$$

The probability that the total claim in the large claims layer exceeds 5 millions can be calculated by looking at the complement, i.e. at the probability that the total claim in the large claims layer does not exceed 5 millions. Since with three claims in the large claims layer we already exceed 5 millions, it is enough to consider only up to two claims. Then we get

$$\begin{aligned}
\mathbb{P}\left[S_{\mathrm{lc}} \leq 5'000'000\right] &= \mathbb{P}[N_{\mathrm{lc}} = 0] + \mathbb{P}[N_{\mathrm{lc}} = 1]\mathbb{P}[Y_{\mathrm{lc}} \leq 5'000'000] + \mathbb{P}[N_{\mathrm{lc}} = 2]\mathbb{P}[Y_{\mathrm{lc}} = 2'000'000]^2 \\
&= \exp\left\{-\frac{\lambda v}{6}\right\} + \exp\left\{-\frac{\lambda v}{6}\right\}\frac{\lambda v}{6}\left(\frac{1}{2}+\frac{1}{4}\right) + \exp\left\{-\frac{\lambda v}{6}\right\}\left(\frac{\lambda v}{6}\right)^2\frac{1}{2}\frac{1}{4} \\
&= \exp\{-0.1\}\left(1 + 0.075 + 0.00125\right) \\
&\approx 97.4\%.
\end{aligned}$$

Hence we can conclude

$$\mathbb{P}\left[S_{\mathrm{lc}} > 5'000'000\right] = 1 - \mathbb{P}\left[S_{\mathrm{lc}} \leq 5'000'000\right] \approx 2.6\%.$$

Solution 4.3 Method of Moments

If $Y \sim \Gamma(\gamma, c)$, then we have

$$\mathbb{E}[Y] = \frac{\gamma}{c} \quad \text{and} \quad \mathrm{Var}(Y) = \frac{\gamma}{c^2}.$$

We define the sample mean $\hat{\mu}_8$ and the sample variance $\hat{\sigma}_8^2$ of the eight observations given on the exercise sheet as

$$\hat{\mu}_8 = \frac{1}{8}\sum_{i=1}^{8} x_i = \frac{64}{8} = 8 \quad \text{and} \quad \hat{\sigma}_8^2 = \frac{1}{7}\sum_{i=1}^{8}(x_i - \hat{\mu}_8)^2 = \frac{28}{7} = 4.$$

The method of moments estimates $(\hat{\gamma}, \hat{c})$ of (γ, c) are defined to be those values that solve the equations

$$\hat{\mu}_8 = \frac{\hat{\gamma}}{\hat{c}} \quad \text{and} \quad \hat{\sigma}_8^2 = \frac{\hat{\gamma}}{\hat{c}^2}.$$

We see that $\hat{\gamma} = \hat{\mu}_8\hat{c}$ and thus

$$\hat{\sigma}_8^2 = \frac{\hat{\mu}_8\hat{c}}{\hat{c}^2} = \frac{\hat{\mu}_8}{\hat{c}},$$

which is equivalent to

$$\hat{c} = \frac{\hat{\mu}_8}{\hat{\sigma}_8^2} = \frac{8}{4} = 2.$$

Moreover, we get

$$\hat{\gamma} = \frac{\hat{\mu}_8^2}{\hat{\sigma}_8^2} = \frac{64}{4} = 16.$$

Thus we conclude that the method of moments estimate are given by $(\hat{\gamma}, \hat{c}) = (16, 2)$.