Non-Life Insurance: Mathematics and Statistics

Solution sheet 6

Solution 6.1 Goodness-of-Fit Test

Let Y be a random variable following a Pareto distribution with threshold $\theta = 200$ and tail index $\alpha = 1.25$. Then the distribution function G of Y is given by

$$G(x) = 1 - \left(\frac{x}{\theta}\right)^{-\alpha} = 1 - \left(\frac{x}{200}\right)^{-1.25}$$

for all $x \geq \theta$. For example for the interval I_2 we then have

$$\mathbb{P}[Y \in I_2] = \mathbb{P}[239 \leq Y < 301] = G(301) - G(239) = 1 - \left(\frac{301}{200}\right)^{-1.25} - \left[1 - \left(\frac{239}{200}\right)^{-1.25}\right] \approx 0.2.$$

By analogous calculations for the other four intervals, we get

$$\mathbb{P}[Y \in I_1] \approx 0.2, \quad \mathbb{P}[Y \in I_2] \approx 0.2, \quad \mathbb{P}[Y \in I_3] \approx 0.2, \quad \mathbb{P}[Y \in I_4] \approx 0.2, \quad \mathbb{P}[Y \in I_5] \approx 0.2.$$

Let E_i and O_i denote respectively the expected number of observations in I_i and the observed number of observations in I_i, for all $i \in \{1, \dots, 5\}$. As we have 20 observations in our data, we can calculate for example E_2 as

$$E_2 = 20 \cdot \mathbb{P}[Y \in I_2] \approx 4.$$

The values of the expected number of observations and the observed number of observations in the five intervals as well as their squared differences are summarized in the following table:

i	1	2	3	4	5
O_i	4	0	8	6	2
E_i	4	4	4	4	4
$(O_i - E_i)^2$	0	16	16	4	4

Now the test statistic of the χ^2-goodness-of-fit test using 5 intervals and 20 observations is given by

$$X^2_{20,5} = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} = \frac{0}{4} + \frac{16}{4} + \frac{16}{4} + \frac{4}{4} + \frac{4}{4} = 10.$$

Let $\alpha = 5\%$. Then the $(1 - \alpha)$-quantile of the χ^2-distribution with $5 - 1 = 4$ degrees of freedom is given by approximately 9.49. Since this is smaller than $X^2_{20,5}$, we can reject the null hypothesis of having a Pareto distribution with threshold $\theta = 200$ and tail index $\alpha = 1.25$ as claim size distribution at the significance level of 5%.

Solution 6.2 Log-Normal Distribution and Deductible

(a) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then the moment generating function M_X of X is given by

$$M_X(r) = \mathbb{E}\left[\exp\{rX\}\right] = \exp\left\{r\mu + \frac{r^2\sigma^2}{2}\right\}$$

for all $r \in \mathbb{R}$. Since Y_1 has a log-normal distribution with mean parameter μ and variance parameter σ^2, we have

$$Y_1 \stackrel{\text{d}}{=} \exp\{X\}.$$

Hence, the expectation, the variance and the coefficient of variation of Y_1 can be calculated as

$$\begin{aligned}
\mathbb{E}[Y_1] &= \mathbb{E}\left[\exp\{X\}\right] = \mathbb{E}\left[\exp\{1 \cdot X\}\right] = M_X(1) = \exp\left\{\mu + \frac{\sigma^2}{2}\right\}, \\
\operatorname{Var}(Y_1) &= \mathbb{E}[Y_1^2] - \mathbb{E}[Y_1]^2 = \mathbb{E}\left[\exp\{2X\}\right] - M_X(1)^2 = M_X(2) - M_X(1)^2 \\
&= \exp\left\{2\mu + \frac{4\sigma^2}{2}\right\} - \exp\left\{2\mu + 2\frac{\sigma^2}{2}\right\} = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\left\{\sigma^2\right\} - 1\right) \quad \text{and} \\
\operatorname{Vco}(Y_1) &= \frac{\sqrt{\operatorname{Var}(Y_1)}}{\mathbb{E}[Y_1]} = \frac{\exp\left\{\mu + \sigma^2/2\right\}\sqrt{\exp\left\{\sigma^2\right\} - 1}}{\exp\left\{\mu + \sigma^2/2\right\}} = \sqrt{\exp\left\{\sigma^2\right\} - 1}.
\end{aligned}$$

(b) From part (a), we know that

$$\begin{aligned}
\sigma &= \sqrt{\log[\operatorname{Vco}(Y_1)^2 + 1]} \quad \text{and} \\
\mu &= \log \mathbb{E}[Y_1] - \frac{\sigma^2}{2}.
\end{aligned}$$

Since $\mathbb{E}[Y_1] = 3\text{'}000$ and $\operatorname{Vco}(Y_1) = 4$, we get

$$\begin{aligned}
\sigma &= \sqrt{\log(4^2 + 1)} \approx 1.68 \quad \text{and} \\
\mu &\approx \log 3\text{'}000 - \frac{(1.68)^2}{2} \approx 6.59.
\end{aligned}$$

(i) The claims frequency λ is given by $\lambda = \mathbb{E}[N]/v$. With the introduction of the deductible $d = 500$, the number of claims changes to

$$N^{\text{new}} = \sum_{i=1}^{N} 1_{\{Y_i > d\}}.$$

Using the independence of N and $Y_1, Y_2, \dots$, we get

$$\mathbb{E}[N^{\text{new}}] = \mathbb{E}\left[\sum_{i=1}^{N} 1_{\{Y_i > d\}}\right] = \mathbb{E}[N]\mathbb{E}[1_{\{Y_1 > d\}}] = \mathbb{E}[N]\mathbb{P}[Y_1 > d].$$

Let Φ denote the distribution function of a standard Gaussian distribution. Since $\log Y_1$ has a Gaussian distribution with mean μ and variance σ^2, we have

$$\mathbb{P}[Y_1 > d] = 1 - \mathbb{P}\left[\frac{\log Y_1 - \mu}{\sigma} \leq \frac{\log d - \mu}{\sigma}\right] = 1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right).$$

Hence, the new claims frequency λ^{new} is given by

$$\lambda^{\text{new}} = \mathbb{E}[N_{\text{new}}]/v = \mathbb{E}[N]\mathbb{P}[Y_1 > d]/v = \lambda \mathbb{P}[Y_1 > d] = \lambda\left[1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)\right].$$

Inserting the values of d, μ and σ, we get

$$\lambda^{\text{new}} \approx \lambda\left[1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)\right] \approx 0.59 \cdot \lambda.$$

Note that the introduction of this deductible reduces the administrative burden a lot, because 41% of (small) claims disappear.

(ii) With the introduction of the deductible $d = 500$, the claim sizes change to

$$Y_i^{\text{new}} = Y_i - d \,|\, Y_i > d.$$

Thus, the new expected claim size is given by

$$\mathbb{E}[Y_1^{\text{new}}] = \mathbb{E}[Y_1 - d | Y_1 > d] = e(d),$$

where $e(d)$ is the mean excess function of Y_1 above d. According to the lecture notes, $e(d)$ is given by

$$e(d) = \mathbb{E}[Y_1] \left[\frac{1 - \Phi\left(\frac{\log d - \mu - \sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)} \right] - d.$$

Inserting the values of d, μ, σ and $\mathbb{E}[Y_1]$, we get

$$\mathbb{E}[Y_1^{\text{new}}] \approx 3'000 \left[\frac{1 - \Phi\left(\frac{\log 500 - 6.59 - 1.68^2}{1.68}\right)}{1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)} \right] - 500 \approx 4'456 \approx 1.49 \cdot \mathbb{E}[Y_1].$$

(iii) According to Proposition 2.2 of the lecture notes, the expected total claim amount $\mathbb{E}[S]$ is given by

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[Y_1].$$

With the introduction of the deductible $d = 500$, the total claim amount S changes to S^{new}, which can be written as

$$S^{\text{new}} = \sum_{i=1}^{N^{\text{new}}} Y_i^{\text{new}}.$$

Hence, the expected total claim amount changes to

$$\begin{aligned}
\mathbb{E}\left[S^{\text{new}}\right] &= \mathbb{E}\left[N^{\text{new}}\right] \mathbb{E}\left[Y_1^{\text{new}}\right] \\
&= \mathbb{E}[N]\mathbb{P}[Y_1 > d] e(d) \\
&= \lambda v \left[1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right) \right] \cdot \left(\mathbb{E}[Y_1] \left[\frac{1 - \Phi\left(\frac{\log d - \mu - \sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)} \right] - d \right).
\end{aligned}$$

Inserting the values of d, μ, σ and $\mathbb{E}[Y_1]$, we get

$$\begin{aligned}
\mathbb{E}\left[S^{\text{new}}\right] &\approx \lambda v \left[1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right) \right] \cdot \left(3'000 \left[\frac{1 - \Phi\left(\frac{\log 500 - 6.59 - 1.68^2}{1.68}\right)}{1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)} \right] - 500 \right) \\
&\approx \lambda v \cdot 0.59 \cdot 4'456 \\
&= 0.88 \cdot \mathbb{E}[S].
\end{aligned}$$

In particular, the insurance company can grant a discount of roughly 12% on the pure risk premium. Note that also the administrative expenses on claims handling will reduce substantially because we only have 59% of the original claims, see the result in (i).

Solution 6.3 Inflation and Deductible

Let Y be a random variable following a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 1$. Then the expectation $\mathbb{E}[Y]$ of Y and the mean excess function $e_Y(u)$ of Y above $u > \theta$ are given by

$$\mathbb{E}[Y] = \frac{\alpha}{\alpha - 1}\theta \quad \text{and} \quad e_Y(u) = \frac{1}{\alpha - 1}u.$$

Since the insurance company only has to pay the part that exceeds the threshold θ, this year's average claim payment z is

$$z = \mathbb{E}[Y] - \theta = \frac{\alpha}{\alpha - 1}\theta - \theta = \frac{\theta}{\alpha - 1}.$$

For the total claim size $\tilde{Y}$ of a claim next year we have

$$\tilde{Y} \stackrel{d}{=} (1 + r)Y \sim \text{Pareto}([1 + r]\theta, \alpha).$$

Let $\rho\theta$ for some $\rho > 0$ denote the increase of the threshold that is needed such that the average claims payment remains unchanged. Then next year's average claim payment is given by

$$\tilde{z} = \mathbb{E}[(\tilde{Y} - [1 + \rho]\theta)_+].$$

Let's first assume that we can choose a $\rho < r$ such that $z = \tilde{z}$. In this case we get

$$\tilde{Y} \geq (1 + r)\theta \quad \text{a.s.} \qquad \Longrightarrow \qquad \tilde{Y} \geq (1 + \rho)\theta \quad \text{a.s.}$$

and thus

$$\tilde{z} = \mathbb{E}[\tilde{Y} - (1 + \rho)\theta] = \mathbb{E}[\tilde{Y}] - (1 + \rho)\theta = \frac{\alpha}{\alpha - 1}(1 + r)\theta - (1 + \rho)\theta.$$

Now we have $z = \tilde{z}$ if and only if

$$\begin{aligned} \frac{\alpha}{\alpha - 1}\theta - \theta &= \frac{\alpha}{\alpha - 1}(1 + r)\theta - (1 + \rho)\theta \\ \Longleftrightarrow \qquad 0 &= \frac{\alpha}{\alpha - 1}r\theta - \rho\theta \\ \Longleftrightarrow \qquad \rho &= \frac{\alpha}{\alpha - 1}r > r, \end{aligned}$$

which is a contradiction to the assumption $\rho < r$. Hence, we conclude that $\rho \geq r$, i.e. the percentage increase in the deductible has to be bigger than the inflation. Assuming $\rho \geq r$, we can calculate

$$\begin{aligned} \tilde{z} &= \mathbb{E}[(\tilde{Y} - [1 + \rho]\theta) \cdot 1_{\{\tilde{Y} - (1+\rho)\theta\}}] \\ &= \mathbb{E}[\tilde{Y} - (1 + \rho)\theta \,|\, \tilde{Y} > (1 + \rho)\theta] \cdot \mathbb{P}[\tilde{Y} > (1 + \rho)\theta] \\ &= e_{\tilde{Y}}([1 + \rho]\theta) \cdot \mathbb{P}[\tilde{Y} > (1 + \rho)\theta] \\ &= \frac{1}{\alpha - 1}(1 + \rho)\theta \cdot \left[\frac{(1 + \rho)\theta}{(1 + r)\theta}\right]^{-\alpha} \\ &= \frac{\theta}{\alpha - 1}(1 + r)^{\alpha}(1 + \rho)^{-\alpha + 1} \\ &= z \cdot (1 + r)^{\alpha}(1 + \rho)^{-\alpha + 1}. \end{aligned}$$

Now we have $z = \tilde{z}$ if and only if

$$(1 + r)^{\alpha}(1 + \rho)^{-\alpha + 1} = 1 \quad \Longleftrightarrow \quad \rho = (1 + r)^{\frac{\alpha}{\alpha - 1}} - 1.$$

We conclude that if we want the claim payment to remain unchanged, we have to increase the deductible θ by the amount

$$\theta\left[(1 + r)^{\frac{\alpha}{\alpha - 1}} - 1\right].$$