Non-Life Insurance: Mathematics and Statistics

Solution sheet 9

Solution 9.1 Utility Indifference Price

(a) Suppose that there exist two utility indifference prices $\pi_1 = \pi_1(u, S, c_0)$ and $\pi_2 = \pi_2(u, S, c_0)$ with $\pi_1 \neq \pi_2$. By definition of a utility indifference price, we have

$$\mathbb{E}[u(c_0 + \pi_1 - S)] = u(c_0) = \mathbb{E}[u(c_0 + \pi_2 - S)]. \tag{1}$$

Without loss of generality, we assume that $\pi_1 < \pi_2$. Then we have

$$c_0 + \pi_1 - S < c_0 + \pi_2 - S \qquad \text{a.s.},$$

which implies

$$u(c_0 + \pi_1 - S) < u(c_0 + \pi_2 - S) \qquad \text{a.s.},$$

since u is a utility function and, thus, strictly increasing by definition. Finally, by taking the expectation, we get

$$\mathbb{E}[u(c_0 + \pi_1 - S)] < \mathbb{E}[u(c_0 + \pi_2 - S)],$$

which is a contradiction to (1). We conclude that if the utility indifference price π exists, then it is unique. Moreover, being a utility function, u is strictly concave by definition. Hence, we can apply Jensen's inequality to get

$$u(c_0) = \mathbb{E}[u(c_0 + \pi - S)] < u(\mathbb{E}[c_0 + \pi - S]) = u(c_0 + \pi - \mathbb{E}[S]).$$

Note that we used that S is non-deterministic and, thus, Jensen's inequality is strict. Since u is strictly increasing, this implies $\pi - \mathbb{E}[S] > 0$, i.e. $\pi > \mathbb{E}[S]$.

(b) Note that

$$\mathbb{E}\left[Y_1^{(1)}\right] = \frac{\gamma}{c} = \frac{20}{0.01} = 2\text{'}000$$

and that

$$\mathbb{E}\left[Y_1^{(2)}\right] = \frac{1}{0.005} = 200.$$

Since S_1 and S_2 both have a compound Poisson distribution, Proposition 2.11 of the lecture notes gives

$$\mathbb{E}[S_1] = \lambda_1 v_1 \mathbb{E}\left[Y_1^{(1)}\right] = \frac{1}{2} \cdot 2\text{'}000 \cdot 2\text{'}000 = 2\text{'}000\text{'}000$$

and

$$\mathbb{E}[S_2] = \lambda_2 v_2 \mathbb{E}\left[Y_1^{(2)}\right] = \frac{1}{10} \cdot 10\text{'}000 \cdot 200 = 200\text{'}000.$$

We conclude that

$$\mathbb{E}[S] = \mathbb{E}[S_1 + S_2] = \mathbb{E}[S_1] + \mathbb{E}[S_2] = 2\text{'}200\text{'}000.$$

(c) The utility indifference price $\pi = \pi(u, S, c_0)$ is defined through the equation

$$u(c_0) = \mathbb{E}[u(c_0 + \pi - S)].$$

Using that the utility function u is given by

$$u(x) = 1 - \frac{1}{\alpha} \exp\{-\alpha x\},$$

for all $x \in \mathbb{R}$, with $\alpha = 1.5 \cdot 10^{-6}$, we get

$$\begin{aligned} u(c_0) = \mathbb{E}[u(c_0 + \pi - S)] &\iff 1 - \frac{1}{\alpha}\exp\{-\alpha c_0\} = \mathbb{E}\left[1 - \frac{1}{\alpha}\exp\{-\alpha(c_0 + \pi - S)\}\right] \\ &\iff \exp\{-\alpha c_0\} = \mathbb{E}\left[\exp\{-\alpha(c_0 + \pi - S)\}\right] \\ &\iff \exp\{\alpha \pi\} = \mathbb{E}\left[\exp\{\alpha S\}\right] \\ &\iff \pi = \frac{1}{\alpha}\log \mathbb{E}\left[\exp\{\alpha S\}\right]. \end{aligned}$$

Note that we can write $S = S_1 + S_2$ and use the independence of S_1 and S_2 to get

$$\begin{aligned} \pi &= \frac{1}{\alpha}\log \mathbb{E}\left[\exp\{\alpha(S_1 + S_2)\}\right] \\ &= \frac{1}{\alpha}\log\left(\mathbb{E}\left[\exp\{\alpha S_1\}\right]\mathbb{E}\left[\exp\{\alpha S_2\}\right]\right) \\ &= \frac{1}{\alpha}\left(\log \mathbb{E}\left[\exp\{\alpha S_1\}\right] + \log \mathbb{E}\left[\exp\{\alpha S_2\}\right]\right) \\ &= \frac{1}{\alpha}\left[\log M_{S_1}(\alpha) + \log M_{S_2}(\alpha)\right], \end{aligned}$$

where M_{S_1} and M_{S_2} denote the moment generating functions of S_1 and S_2, respectively. Moreover, since S_1 and S_2 both have a compound Poisson distribution, Proposition 2.11 of the lecture notes gives

$$\pi = \frac{1}{\alpha}\left(\lambda_1 v_1 \left[M_{Y_1^{(1)}}(\alpha) - 1\right] + \lambda_2 v_2 \left[M_{Y_1^{(2)}}(\alpha) - 1\right]\right),$$

where $M_{Y_1^{(1)}}$ and $M_{Y_1^{(2)}}$ denote the moment generating functions of $Y_1^{(1)}$ and $Y_1^{(2)}$, respectively. Using that $Y_1^{(1)} \sim \Gamma(\gamma = 20, c = 0.01)$ and that $Y_1^{(2)} \sim \mathrm{expo}(0.005)$, we get

$$M_{Y_1^{(1)}}(\alpha) = \left(\frac{c}{c - \alpha}\right)^{\gamma} = \left(\frac{0.01}{0.01 - 1.5 \cdot 10^{-6}}\right)^{20}$$

and

$$M_{Y_1^{(2)}}(\alpha) = \frac{0.005}{0.005 - \alpha} = \frac{0.005}{0.005 - 1.5 \cdot 10^{-6}}.$$

In particular, since $\alpha < c$ and $\alpha < 0.005$, both $M_{Y_1^{(1)}}(\alpha)$ and $M_{Y_1^{(2)}}(\alpha)$ and thus also $M_{S_1}(\alpha)$ and $M_{S_2}(\alpha)$ exist. Inserting all the numerical values, we find the utility indifference price

$$\begin{aligned} \pi &= \frac{2}{3} \cdot 10^6 \left(\frac{1}{2} \cdot 2'000 \cdot \left[\left(\frac{0.01}{0.01 - 1.5 \cdot 10^{-6}}\right)^{20} - 1\right] + \frac{1}{10} \cdot 10'000 \cdot \left[\frac{0.005}{0.005 - 1.5 \cdot 10^{-6}} - 1\right]\right) \\ &= 2'203'213. \end{aligned}$$

Note that we have

$$\frac{\pi - \mathbb{E}[S]}{\mathbb{E}[S]} = \frac{2'203'213 - 2'200'000}{2'200'000} = \frac{3'213}{2'200'000} \approx 0.146\%.$$

Thus, the loading $\pi - \mathbb{E}[S]$ is given by approximately 0.146% of the pure risk premium.

(d) The moment generating function M_X of $X \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ is given by

$$M_X(r) = \exp\left\{r\mu + \frac{r^2\sigma^2}{2}\right\},$$

for all $r \in \mathbb{R}$. Hence, if we assume Gaussian distributions for S_1 and S_2, then we get

$$\begin{aligned}
\pi &= \frac{1}{\alpha}\left[\log M_{S_1}(\alpha) + \log M_{S_2}(\alpha)\right] \\
&= \frac{1}{\alpha}\left(\alpha\mathbb{E}[S_1] + \frac{\alpha^2}{2}\mathrm{Var}(S_1) + \alpha\mathbb{E}[S_2] + \frac{\alpha^2}{2}\mathrm{Var}(S_2)\right) \\
&= \mathbb{E}[S_1] + \mathbb{E}[S_2] + \frac{\alpha}{2}[\mathrm{Var}(S_1) + \mathrm{Var}(S_2)] \\
&= \mathbb{E}[S] + \frac{\alpha}{2}\mathrm{Var}(S),
\end{aligned}$$

where in the last equation we used that S_1 and S_2 are independent. We see that in this case the utility indifference price is given according to a variance loading principle. Since here we assume Gaussian distributions for S_1 and S_2 with the same corresponding first two moments as in the compound Poisson case in part (c), in order to calculate $\mathrm{Var}(S_1)$ and $\mathrm{Var}(S_2)$, we again assume that S_1 and S_2 have compound Poisson distributions. Note that

$$\mathbb{E}\left[(Y_1^{(1)})^2\right] = \frac{\gamma(\gamma+1)}{c^2} = \frac{20 \cdot 21}{0.01^2} = 4\text{'}200\text{'}000,$$

and that

$$\mathbb{E}\left[(Y_1^{(2)})^2\right] = \frac{2}{0.005^2} = 80\text{'}000.$$

Then Proposition 2.11 of the lecture notes gives

$$\mathrm{Var}(S_1) = \lambda_1 v_1 \mathbb{E}\left[(Y_1^{(1)})^2\right] = \frac{1}{2} \cdot 2\text{'}000 \cdot 4\text{'}200\text{'}000 = 4\text{'}200\text{'}000\text{'}000$$

and

$$\mathrm{Var}(S_2) = \lambda_2 v_2 \mathbb{E}\left[(Y_1^{(2)})^2\right] = \frac{1}{10} \cdot 10\text{'}000 \cdot 80\text{'}000 = 80\text{'}000\text{'}000,$$

which leads to

$$\mathrm{Var}(S) = \mathrm{Var}(S_1 + S_2) = \mathrm{Var}(S_1) + \mathrm{Var}(S_2) = 4\text{'}280\text{'}000\text{'}000.$$

We conclude that the utility indifference price is given by

$$\pi = \mathbb{E}[S] + \frac{\alpha}{2}\mathrm{Var}(S) = 2\text{'}200\text{'}000 + \frac{1.5 \cdot 10^{-6}}{2} \cdot 4\text{'}280\text{'}000\text{'}000 = 2\text{'}203\text{'}210.$$

Note that we have

$$\frac{\pi - \mathbb{E}[S]}{\mathbb{E}[S]} = \frac{2\text{'}203\text{'}210 - 2\text{'}200\text{'}000}{2\text{'}200\text{'}000} = \frac{3\text{'}210}{2\text{'}200\text{'}000} \approx 0.146\%.$$

Thus, as in part (c), the loading $\pi - \mathbb{E}[S]$ is given by approximately 0.146% of the pure risk premium. The reason why we get the same results in (c) and (d) is the Central Limit Theorem. In particular, neither the gamma distribution nor the exponential distribution are heavy-tailed distributions and thus $\lambda_1 v_1 = \lambda_2 v_2 = 1\text{'}000$ are large enough for the normal approximations to be valid approximations for the compound Poisson distributions.

Solution 9.2 Value-at-Risk and Expected Shortfall

(a) Since $S \sim \mathrm{LN}(\mu, \sigma^2)$ with $\mu = 20$ and $\sigma^2 = 0.015$, we have

$$\mathbb{E}[S] = \exp\left\{\mu + \frac{\sigma^2}{2}\right\} \approx 488'817'614.$$

Let z denote the VaR of $S - \mathbb{E}[S]$ at security level $1 - q = 99.5\%$. Then, since the distribution function of a lognormal distribution is continuous and strictly increasing, z is defined via the equation

$$\mathbb{P}[S - \mathbb{E}[S] \leq z] = 1 - q.$$

By writing Φ for the distribution function of a standard Gaussian distribution, we can calculate z as follows

$$\begin{aligned}
\mathbb{P}[S - \mathbb{E}[S] \leq z] = 1 - q \quad &\Longleftrightarrow \quad \mathbb{P}[S \leq z + \mathbb{E}[S]] = 1 - q \\
&\Longleftrightarrow \quad \mathbb{P}\left[\frac{\log S - \mu}{\sigma} \leq \frac{\log(z + \mathbb{E}[S]) - \mu}{\sigma}\right] = 1 - q \\
&\Longleftrightarrow \quad \Phi\left[\frac{\log(z + \mathbb{E}[S]) - \mu}{\sigma}\right] = 1 - q \\
&\Longleftrightarrow \quad \log(z + \mathbb{E}[S]) = \mu + \sigma \cdot \Phi^{-1}(1 - q) \\
&\Longleftrightarrow \quad z = \exp\left\{\mu + \sigma \cdot \Phi^{-1}(1 - q)\right\} - \mathbb{E}[S] \\
&\Longleftrightarrow \quad z = \exp\{\mu\} \left(\exp\{\sigma \cdot \Phi^{-1}(1 - q)\} - \exp\left\{\frac{\sigma^2}{2}\right\}\right).
\end{aligned}$$

For $1 - q = 99.5\%$, we have $\Phi^{-1}(1 - q) \approx 2.576$. Thus, we get

$$z \approx 176'299'286.$$

In particular, π_{CoC} is then given by

$$\pi_{\mathrm{CoC}} = \mathbb{E}[S] + r_{\mathrm{CoC}} \cdot z \approx 488'817'614 + 0.06 \cdot 176'299'286 \approx 499'395'571.$$

Note that we have

$$\frac{\pi_{\mathrm{CoC}} - \mathbb{E}[S]}{\mathbb{E}[S]} \approx \frac{499'395'571 - 488'817'614}{488'817'614} = \frac{10'577'957}{488'817'614} \approx 2.164\%.$$

Thus, the loading $\pi_{\mathrm{CoC}} - \mathbb{E}[S]$ is given by approximately 2.164% of the pure risk premium.

(b) For all $u \in (0, 1)$, let VaR_u and ES_u denote the VaR risk measure and the expected shortfall risk measure, respectively, at security level u. Note that actually in part (a) we found that

$$\mathrm{VaR}_u(S - \mathbb{E}[S]) = \exp\left\{\mu + \sigma \cdot \Phi^{-1}(u)\right\} - \mathbb{E}[S]$$

and that by a similar computation we get

$$\mathrm{VaR}_u(S) = \exp\left\{\mu + \sigma \cdot \Phi^{-1}(u)\right\},$$

for all $u \in (0, 1)$. In particular, we have

$$\mathrm{VaR}_u(S - \mathbb{E}[S]) + \mathbb{E}[S] = \mathrm{VaR}_u(S)$$

for all $u \in (0,1)$. Since the distribution function of S is continuous and strictly increasing, according to Example 6.26 of the lecture notes we have

$$\begin{aligned}
\mathrm{ES}_{1-q}(S-\mathbb{E}[S]) &= \mathbb{E}\left[S-\mathbb{E}[S] \,|\, S-\mathbb{E}[S] \geq \mathrm{VaR}_{1-q}(S-\mathbb{E}[S])\right] \\
&= \mathbb{E}\left[S-\mathbb{E}[S] \,|\, S \geq \mathrm{VaR}_{1-q}(S)\right] \\
&= \mathbb{E}\left[S \,|\, S \geq \mathrm{VaR}_{1-q}(S)\right] - \mathbb{E}[S] \\
&= \mathrm{ES}_{1-q}(S) - \mathbb{E}[S].
\end{aligned}$$

By definition of the mean excess function $e_S(\cdot)$ of S we have

$$\mathrm{ES}_{1-q}(S) = \mathbb{E}\left[S-\mathrm{VaR}_{1-q}(S) \,|\, S \geq \mathrm{VaR}_{1-q}(S)\right]+\mathrm{VaR}_{1-q}(S) = e_S[\mathrm{VaR}_{1-q}(S)]+\mathrm{VaR}_{1-q}(S).$$

Moreover, according to the formula given in Chapter 3.2.3 of the lecture notes, the mean excess function $e_S[\mathrm{VaR}_{1-q}(S)]$ above level $\mathrm{VaR}_{1-q}(S)$ is given by

$$e_S[\mathrm{VaR}_{1-q}(S)] = \mathbb{E}[S]\left(\frac{1-\Phi\left[\frac{\log \mathrm{VaR}_{1-q}(S)-\mu-\sigma^2}{\sigma}\right]}{1-\Phi\left[\frac{\log \mathrm{VaR}_{1-q}(S)-\mu}{\sigma}\right]}\right) - \mathrm{VaR}_{1-q}(S).$$

Using the formula calculated above for $\mathrm{VaR}_u(S)$ with $u = 1-q$, we get

$$\begin{aligned}
\mathrm{ES}_{1-q}(S) &= \mathbb{E}[S]\left(\frac{1-\Phi\left[\frac{\log \mathrm{VaR}_{1-q}(S)-\mu-\sigma^2}{\sigma}\right]}{1-\Phi\left[\frac{\log \mathrm{VaR}_{1-q}(S)-\mu}{\sigma}\right]}\right) \\
&= \mathbb{E}[S]\left(\frac{1-\Phi\left[\frac{\mu+\sigma\cdot\Phi^{-1}(1-q)-\mu-\sigma^2}{\sigma}\right]}{1-\Phi\left[\frac{\mu+\sigma\cdot\Phi^{-1}(1-q)-\mu}{\sigma}\right]}\right) \\
&= \mathbb{E}[S]\left(\frac{1-\Phi\left[\Phi^{-1}(1-q)-\sigma\right]}{1-\Phi\left[\Phi^{-1}(1-q)\right]}\right) \\
&= \mathbb{E}[S]\,\frac{1}{q}\,\left(1-\Phi\left[\Phi^{-1}(1-q)-\sigma\right]\right).
\end{aligned}$$

In particular, we have found

$$\begin{aligned}
\mathrm{ES}_{1-q}(S-\mathbb{E}[S]) &= \frac{1}{q}\,\mathbb{E}[S]\left(1-\Phi\left[\Phi^{-1}(1-q)-\sigma\right]\right) - \mathbb{E}[S] \\
&= \frac{1}{q}\,\mathbb{E}[S]\left(1-q-\Phi\left[\Phi^{-1}(1-q)-\sigma\right]\right) \\
&= \frac{1}{q}\,\exp\left\{\mu+\frac{\sigma^2}{2}\right\}\left(1-q-\Phi\left[\Phi^{-1}(1-q)-\sigma\right]\right).
\end{aligned}$$

For $1-q = 99\%$, we get

$$\mathrm{ES}_{99\%}(S-\mathbb{E}[S]) \approx 184\text{'}119\text{'}256.$$

Finally, π_{CoC} is then given by

$$\pi_{\mathrm{CoC}} = \mathbb{E}[S] + r_{\mathrm{CoC}}\cdot\mathrm{ES}_{99\%}(S-\mathbb{E}[S]) \approx 488\text{'}817\text{'}614 + 0.06\cdot 184\text{'}119\text{'}256 \approx 499\text{'}864\text{'}769.$$

Note that we have

$$\frac{\pi_{\mathrm{CoC}}-\mathbb{E}[S]}{\mathbb{E}[S]} \approx \frac{499\text{'}864\text{'}769 - 488\text{'}817\text{'}614}{488\text{'}817\text{'}614} = \frac{11\text{'}047\text{'}155}{488\text{'}817\text{'}614} \approx 2.26\%.$$

Thus, the loading $\pi_{\mathrm{CoC}} - \mathbb{E}[S]$ is given by approximately 2.26% of the pure risk premium. In particular, the cost-of-capital price in this example is higher using the expected shortfall risk measure at security level 99% than using the VaR risk measure at security level 99.5%.

(c) In parts (a) and (b) we found that

$$\mathrm{VaR}_{99.5\%}(S - \mathbb{E}[S]) < \mathrm{ES}_{99\%}(S - \mathbb{E}[S]).$$

Let $1 - q = 99\%$. Now the goal is to find $u \in [0, 1]$ such that

$$\mathrm{VaR}_u(S - \mathbb{E}[S]) = \mathrm{ES}_{1-q}(S - \mathbb{E}[S]) \quad \Longleftrightarrow \quad \mathrm{VaR}_u(S) = \mathrm{ES}_{1-q}(S).$$

Note that from part (b) we know

$$\mathrm{VaR}_u(S) = \exp\left\{\mu + \sigma \cdot \Phi^{-1}(u)\right\},$$

for all $u \in (0, 1)$, and

$$\mathrm{ES}_{1-q}(S) = \frac{1}{q}\,\mathbb{E}[S]\left(1 - \Phi\left[\Phi^{-1}(1-q) - \sigma\right]\right).$$

Hence, we can solve for u to get

$$\begin{aligned} u &= \Phi\left(\frac{\log\left[\frac{1}{q}\,\mathbb{E}[S]\left(1 - \Phi\left[\Phi^{-1}(1-q) - \sigma\right]\right)\right] - \mu}{\sigma}\right) \\ &\approx 99.62\%. \end{aligned}$$

We conclude that in this example the cost-of-capital price using the VaR risk measure at security level 99.62% is approximately equal to the cost-of-capital price using the expected shortfall risk measure at security level 99%.

(d) Since $S \sim \mathrm{LN}(\mu, \sigma^2)$ with $\mu = 20$ and $\sigma^2 = 0.015$ and U and V are assumed to be independent, we have

$$U \sim \mathcal{N}(\mu, \sigma^2), \quad V \sim \mathcal{N}(\mu, \sigma^2) \quad \text{and} \quad U + V \sim \mathcal{N}(2\mu, 2\sigma^2).$$

Let $X \sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2)$ for some $\tilde{\mu} \in \mathbb{R}$ and $\tilde{\sigma}^2 > 0$. Then $\mathrm{VaR}_{1-q}(X)$ can be calculated as

$$\begin{aligned} \mathbb{P}\left[X \le \mathrm{VaR}_{1-q}(X)\right] = 1 - q \quad &\Longleftrightarrow \quad \mathbb{P}\left[\frac{X - \tilde{\mu}}{\tilde{\sigma}} \le \frac{\mathrm{VaR}_{1-q}(X) - \tilde{\mu}}{\tilde{\sigma}}\right] = 1 - q \\ &\Longleftrightarrow \quad \Phi\left[\frac{\mathrm{VaR}_{1-q}(X) - \tilde{\mu}}{\tilde{\sigma}}\right] = 1 - q \\ &\Longleftrightarrow \quad \mathrm{VaR}_{1-q}(X) = \tilde{\mu} + \tilde{\sigma} \cdot \Phi^{-1}(1-q). \end{aligned}$$

This implies that

$$\mathrm{VaR}_{1-q}(U) + \mathrm{VaR}_{1-q}(V) = \mu + \sigma \cdot \Phi^{-1}(1-q) + \mu + \sigma \cdot \Phi^{-1}(1-q) = 2\mu + 2\sigma \cdot \Phi^{-1}(1-q)$$

and that

$$\mathrm{VaR}_{1-q}(U + V) = 2\mu + \sqrt{2}\sigma \cdot \Phi^{-1}(1-q).$$

Since $\Phi^{-1}(0.45) \approx -0.126$ and $\Phi^{-1}(0.55) \approx 0.126$, we get

$$\mathrm{VaR}_{0.45}(U + V) \approx 39.978 > 39.969 \approx \mathrm{VaR}_{0.45}(U) + \mathrm{VaR}_{0.45}(V)$$

and

$$\mathrm{VaR}_{0.55}(U + V) \approx 40.022 < 40.031 \approx \mathrm{VaR}_{0.55}(U) + \mathrm{VaR}_{0.55}(V).$$

Note that since

$$\begin{aligned} \mathrm{VaR}_{1-q}(U + V) > \mathrm{VaR}_{1-q}(U) + \mathrm{VaR}_{1-q}(V) \quad &\Longleftrightarrow \quad \Phi^{-1}(1-q) > \sqrt{2}\Phi^{-1}(1-q) \\ &\Longleftrightarrow \quad \Phi^{-1}(1-q) < 0, \end{aligned}$$

one can see that in this example

$$\mathrm{VaR}_{1-q}(U+V) > \mathrm{VaR}_{1-q}(U) + \mathrm{VaR}_{1-q}(V)$$

for all $1-q \in \left(0, \frac{1}{2}\right)$ and that

$$\mathrm{VaR}_{1-q}(U+V) < \mathrm{VaR}_{1-q}(U) + \mathrm{VaR}_{1-q}(V)$$

for all $1-q \in \left(\frac{1}{2}, 1\right)$.

Solution 9.3 Esscher Premium

(a) Let $\alpha \in (0, r_0)$ and M_S' and M_S'' denote the first and second derivative of M_S, respectively. According to the proof of Corollary 6.16 of the lecture notes, the Esscher premium π_α can be written as

$$\pi_\alpha = \frac{M_S'(\alpha)}{M_S(\alpha)}.$$

Hence, the derivative of π_α can be calculated as

$$\begin{aligned}
\frac{d}{d\alpha}\pi_\alpha &= \frac{d}{d\alpha}\frac{M_S'(\alpha)}{M_S(\alpha)} \\
&= \frac{M_S''(\alpha)}{M_S(\alpha)} - \left(\frac{M_S'(\alpha)}{M_S(\alpha)}\right)^2 \\
&= \frac{\mathbb{E}\left[S^2 \exp\{\alpha S\}\right]}{M_S(\alpha)} - \left(\frac{\mathbb{E}\left[S \exp\{\alpha S\}\right]}{M_S(\alpha)}\right)^2 \\
&= \frac{1}{M_S(\alpha)} \int_{-\infty}^{\infty} x^2 \exp\{\alpha x\}\, dF(x) - \left[\frac{1}{M_S(\alpha)} \int_{-\infty}^{\infty} x \exp\{\alpha x\}\, dF(x)\right]^2 \\
&= \int_{-\infty}^{\infty} x^2\, dF_\alpha(x) - \left[\int_{-\infty}^{\infty} x\, dF_\alpha(x)\right]^2,
\end{aligned}$$

where we define the distribution function F_α by

$$F_\alpha(s) = \frac{1}{M_S(\alpha)} \int_{-\infty}^{s} \exp\{\alpha x\}\, dF(x),$$

for all $s \in \mathbb{R}$. Let X be a random variable with distribution function F_α. Then we get

$$\frac{d}{d\alpha}\pi_\alpha = \int_{-\infty}^{\infty} x^2\, dF_\alpha(x) - \left[\int_{-\infty}^{\infty} x\, dF_\alpha(x)\right]^2 = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = \mathrm{Var}(X) \geq 0.$$

Hence, the Esscher premium π_α is always non-decreasing. Moreover, if S is non-deterministic, then also X is non-deterministic. Thus, in this case we get

$$\frac{d}{d\alpha}\pi_\alpha = \mathrm{Var}(X) > 0.$$

In particular, the Esscher premium π_α then is strictly increasing in α.

(b) Let $\alpha \in (0, r_0)$. According to Corollary 6.16 of the lecture notes, the Esscher premium π_α is given by

$$\pi_\alpha = \frac{d}{dr}\log M_S(r)\bigg|_{r=\alpha}.$$

For small values of α, we can use a first-order Taylor approximation around 0 to get

$$\begin{aligned}
\pi_\alpha &\approx \frac{d}{dr}\log M_S(r)\bigg|_{r=0} + \alpha\cdot\frac{d^2}{dr^2}\log M_S(r)\bigg|_{r=0}\\
&= \frac{M_S'(0)}{M_S(0)} + \alpha\left(\frac{M_S''(0)}{M_S(0)} - \left[\frac{M_S'(0)}{M_S(0)}\right]^2\right)\\
&= \mathbb{E}[S] + \alpha\left(\mathbb{E}\left[S^2\right] - \mathbb{E}[S]^2\right)\\
&= \mathbb{E}[S] + \alpha\mathrm{Var}(S).
\end{aligned}$$

We conclude that for small values of α, the Esscher premium π_α of S is approximately equal to a premium resulting from a variance loading principle.

(c) Since $S \sim \mathrm{CompPoi}(\lambda v, G)$, we can use Proposition 2.11 of the lecture notes to get

$$\log M_S(r) = \lambda v\left[M_G(r) - 1\right],$$

where M_G denotes the moment generating function of a random variable with distribution function G. Since G is the distribution function of a gamma distribution with shape parameter $\gamma > 0$ and scale parameter $c > 0$, we have

$$M_G(r) = \left(\frac{c}{c-r}\right)^\gamma,$$

for all $r < c$. In particular, also $M_S(r)$ is defined for all $r < c$, which implies that the Esscher premium π_α exists for all $\alpha \in (0, c)$.

Now let $\alpha \in (0, c)$. Then the Esscher premium π_α can be calculated as

$$\begin{aligned}
\pi_\alpha &= \frac{d}{dr}\log M_S(r)\bigg|_{r=\alpha}\\
&= \frac{d}{dr}\lambda v\left[\left(\frac{c}{c-r}\right)^\gamma - 1\right]\bigg|_{r=\alpha}\\
&= \frac{d}{dr}\lambda v\left[\left(1-\frac{r}{c}\right)^{-\gamma} - 1\right]\bigg|_{r=\alpha}\\
&= \lambda v\frac{\gamma}{c}\left(1-\frac{r}{c}\right)^{-\gamma-1}\bigg|_{r=\alpha}\\
&= \lambda v\frac{\gamma}{c}\left(\frac{c}{c-\alpha}\right)^{\gamma+1}.
\end{aligned}$$

Note that since $c > c - \alpha$ and $\gamma > 0$, we have

$$\left(\frac{c}{c-\alpha}\right)^{\gamma+1} > 1,$$

and, thus,

$$\pi_\alpha = \lambda v\frac{\gamma}{c}\left(\frac{c}{c-\alpha}\right)^{\gamma+1} > \lambda v\frac{\gamma}{c} = \mathbb{E}[S].$$