D-MATH
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Algebraic Geometry II

HS 2017

Exercise Sheet 1

$\mathcal{O}_X$-Modules, Quasi-coherent Sheaves

General Rule: We recommend that you do, or at least try to, solve all unmarked problems. Problems marked $*$ are additional ones that may be harder or may lead into directions not immediately covered by the course. Problems marked $**$ are challenge problems; if you try them, discuss your results with Prof. Pink.

Fix a scheme $(X, \mathcal{O}_X)$.

1. Explain in all details why and how the tensor product of sheaves of $\mathcal{O}_X$-modules is functorial in both variables.

2. (*Basic properties of the sheaf of homomorphisms*) Consider $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{F}_i$. In each of the following cases construct a natural isomomorphism, or a natural homomorphism and discuss under which additional conditions this is an isomorphism.

   (a) Between $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ and $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ for any $x \in X$.

   (b) Between $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$ and $\mathcal{F}$.

   (c) Between $\mathscr{H}om_{\mathcal{O}_X}(\bigoplus_{i\in I} \mathcal{F}_i, \mathcal{G})$ and $\prod_{i\in I} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{G})$.

   (d) Between $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \prod_{i\in I} \mathcal{F}_i)$ and $\prod_{i\in I} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}_i)$.

   (e) Between $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H})$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$.

   (f) Between $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H})$ and $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$.

3. (*Exactness properties of* Hom *and* $\mathscr{H}om$) Prove the following:

   (a) A sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ of $\mathcal{O}_X$-modules is exact if and only if for all open subsets $U \subset X$ and for all $\mathcal{O}_U$-modules $\mathcal{G}$ the sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}'|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}''|_U)$$

   of $\mathcal{O}_X(U)$-modules is exact if and only if for all $\mathcal{O}_X$-modules $\mathcal{G}$ the sequence

$$0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}') \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}'')$$

   of $\mathcal{O}_X$-modules is exact.

(b) A sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of $\mathcal{O}_X$-modules is exact if and only if for all open subsets $U \subset X$ and for all $\mathcal{O}_U$-modules $\mathcal{G}$ the sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}''|_U, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}'|_U, \mathcal{G})$$

of $\mathcal{O}_X(U)$-modules is exact if and only if for all $\mathcal{O}_X$-modules $\mathcal{G}$ the sequence

$$0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathscr{H}om_{\mathcal{O}}(\mathcal{F}', \mathcal{G})$$

of $\mathcal{O}_X$-modules is exact.

4. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank.

   (a) Show that $(\mathcal{E}^\vee)^\vee \cong \mathcal{E}$.

   (b) For any $\mathcal{O}_X$-module $\mathcal{F}$, show that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes \mathcal{F}$.

   (c) Let n be the rank of $\mathcal{E}$ and suppose that $\mathcal{F}$ is locally free of finite rank m. Show that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is locally free of rank nm.

5. (a) If $X = \operatorname{Spec} A$, show that for any A-modules M and N, **where M is finitely presented**, there is an isomorphism $\mathrm{Hom}_A(M, N)^\sim \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$ of $\mathcal{O}_X$-modules which is functorial in M and N.

   (b) Show that for any quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, **where $\mathcal{F}$ is finitely presented**, the $\mathcal{O}_X$-module $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is again quasi-coherent.

**6. Give an example of a scheme X and a set I such that $\mathcal{O}_X^I$ is not free.