Exercise Sheet 10

Cohomology, Ampleness Criterion, Higher Direct Image

1. Let X be a projective scheme over a noetherian ring A. Consider a finite exact sequence $\mathcal{F}^1 \to \mathcal{F}^2 \to \ldots \to \mathcal{F}^r$ of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geqslant n_0$, the sequence of global sections

$$\mathcal{F}^1(n)(X) \to \mathcal{F}^2(n)(X) \to \ldots \to \mathcal{F}^r(n)(X)$$

is exact.

2. Consider separated quasicompact morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and a quasicoherent sheaf $\mathcal{F}$ on X. Prove that there are natural isomorphisms

$$R^p(g \circ f)_*\mathcal{F} \;\cong\; \begin{cases} R^p g_*(f_*\mathcal{F}) & \text{if } f \text{ is affine,} \\ g_*(R^p f_*\mathcal{F}) & \text{if } g \text{ is affine.} \end{cases}$$

3. Let X and Y be proper schemes over a noetherian ring A. Consider an invertible sheaf $\mathcal{L}$ on X. Prove:

   (a) If $\mathcal{L}$ is ample and $i\colon Y \hookrightarrow X$ is any closed embedding, then $i^*\mathcal{L}$ is ample.

   (b) The sheaf $\mathcal{L}$ is ample if and only if $i^*\mathcal{L}$ is ample for every reduced irreducible component $i\colon Y \hookrightarrow X$.

   (c) For any finite surjective morphism $f\colon Y \to X$, the sheaf $\mathcal{L}$ is ample if an only if $f^*\mathcal{L}$ is ample.

4. Let $X := \mathbb{P}^n_k$ for a field k.

   (a) Show that for any integer $0 \leqslant q \leqslant n$ there is a short exact sequence

$$0 \to \Omega^q_{X/k} \to \mathcal{O}_X(-q)^{\binom{n+1}{q}} \to \Omega^{q-1}_{X/k} \to 0.$$

   (b) Compute $\dim_k H^p(X, \Omega^q_{X/k})$ for $X := \mathbb{P}^n_k$ and all p, q.

5. Let Y be the curve in $\mathbb{P}^3_k$ over a field k that is defined by the equations $X_0^2 + X_2^2 = aX_1X_3$ and $X_1^2 + X_3^2 = aX_0X_2$ for a constant $a \in k$. Compute $H^*(Y, \mathcal{O}_Y)$, find out when Y is regular, and in that case compute $H^*(Y, \Omega_{Y/k})$.

*6. (a) Let $(C^\bullet, d_C)$ and $(D^\bullet, d_D)$ be complexes of A-modules, where A is a ring. We define a complex $(C \otimes D)^\bullet$ whose degree m part is given by

$$(C \otimes D)^m = \bigoplus_{p+q=m} C^p \otimes_A D^q$$

for every $m \in \mathbb{Z}$. The boundary maps d are given on each summand via $d(f \otimes g) := d_C f \otimes g + (-1)^{\deg f} f \otimes d_D g$, and we extend by linearity. We call $\big((C \otimes D)^\bullet, d\big)$ the *tensor product* of $C^\bullet$ and $D^\bullet$. Show that if A is a field, we have a natural isomorphism

$$H^m\big((C \otimes D)^\bullet\big) \cong \bigoplus_{p+q=m} H^p(C^\bullet) \otimes_A H^q(D^\bullet).$$

(b) (*Künneth Formula.*) Let X and Y be quasi-compact separated schemes over a field k, and let $\mathcal{F}$ and $\mathcal{G}$ be quasicoherent sheaves on X and Y respectively. Show that for every $m \geqslant 0$, there is a natural isomorphism

$$H^m(X \times_k Y, \mathrm{pr}_X^* \mathcal{F} \otimes \mathrm{pr}_Y^* \mathcal{G}) \cong \bigoplus_{p+q=m} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}).$$