D-MATH | Algebraic Geometry II | HS 2017
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Exercise Sheet 11

Higher Direct Image, Duality, Base Change

1. Let $f\colon X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{L}$ be a relatively ample invertible sheaf on X over Y, and let $\mathcal{F}$ be a coherent sheaf on X. Show:

   (a) For all $n \gg 0$, the natural map $f^*f_*\big(\mathcal{F} \otimes \mathcal{L}^{\otimes n}\big) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is surjective.

   (b) For $p > 0$ and $n \gg 0$, we have $R^pf_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$.

2. Show the following:

   (a) For any flat morphism $f\colon X \to Y$ the functor f^* from the category of $\mathcal{O}_Y$-modules to the category of $\mathcal{O}_X$-modules is exact.

   (b) For any morphism $f\colon X \to Y$ and any flat $\mathcal{O}_Y$-module $\mathcal{G}$ the $\mathcal{O}_X$-module $f^*\mathcal{G}$ is flat.

   (c) For any flat morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ the composite $g \circ f$ is flat.

   (d) For any flat morphism $f\colon X \to Y$ and any morphism $g\colon Y' \to Y$ the morphism $X \times_Y Y' \to Y'$ is flat.

**3. Show that every smooth morphism is flat.

4. Let $f\colon X \to Y$ be a projective morphism with r-dualizing sheaf ω_f. Show that for any flat morphism $Y' \to Y$, the dualizing sheaf of $X \times_Y Y' \to Y'$ is isomorphic to $\mathrm{pr}_X^*\omega_f$.

5. (*Projection Formula*) (Compare Sheet 3, Exercise 2) Consider a morphism $f\colon X \to Y$, an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$.

   (a) Construct a natural base change homomorphism $(R^pf_*\mathcal{F}) \otimes \mathcal{G} \to R^pf_*(\mathcal{F} \otimes f^*\mathcal{G})$.

   (b) If f is separated and quasi-compact and $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent and $\mathcal{G}$ is flat, then this is an isomorphism.

6. Let $Y = \operatorname{Spec} A$ and $X = \operatorname{Proj} A[X,Y] = \mathbb{P}^1_A$. Let $\mathcal{F}$ be the kernel of the homomorphism $\varphi := (X^2, aXY, Y^2)\colon \mathcal{O}_X^{\oplus 3} \to \mathcal{O}_X(2)$ for some $a \in A$.

   (a) Show that $\mathcal{F}$ is locally free and the sequence $0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus 3} \xrightarrow{\varphi} \mathcal{O}_X(2) \to 0$ is exact.

   (b) For every integer p compute $H^p(X, \mathcal{F})$.

   (c) For every point $y \in Y$ with fiber $i_y\colon X_y \hookrightarrow X$ and every integer p compute $H^p(X_y, i_y^*\mathcal{F})$ and compare it with $H^p(X, \mathcal{F}) \otimes_A k(y)$.