D-MATH
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Algebraic Geometry II

HS 2017

Exercise Sheet 12

Euler Characteristic, Riemann-Roch, Residues

1. (*Riemann-Roch for locally free sheaves*) Let X be a connected smooth projective curve of genus g over an algebraically closed field k.

   (a) For every non-zero locally free sheaf $\mathcal{F}$ there exists an invertible sheaf $\mathcal{L} \subset \mathcal{F}$ such that $\mathcal{F}/\mathcal{L}$ is locally free.

   (b) For any locally free sheaf $\mathcal{F}$ of rank r over X define $\deg(\mathcal{F}) := \deg(\bigwedge^r \mathcal{F})$ and prove that
   $$\chi(X, \mathcal{F}) \;=\; r \cdot (1-g) + \deg(\mathcal{F}).$$

2. For an arbitrary integral projective curve X over an algebraically closed field k, the *arithmetic genus of* X is defined as $p_a(Y) := h^1(X, \mathcal{O}_X)$. Let $\pi\colon \tilde{X} \to X$ be the normalization of X.

   (a) Show that $p_a(X) = p_a(\tilde{X}) + \sum'_{P \in X} \mathrm{length}_{\mathcal{O}_{X,P}}(\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)_P$.

   (b) Deduce that $p_a(X) = 0$ if and only if X is nonsingular of genus 0.

   (c) Determine $p_a(X)$ for the nodal cubic curve $X := V(C(C-B)A - B^3) \subset \mathbb{P}^2_k$ and the cuspidal cubic curve $X := V(B^2C - A^3) \subset \mathbb{P}^2_k$.

3. (*Hilbert polynomial of a coherent sheaf*) Let X be a projective scheme over a field k with a very ample invertible sheaf $\mathcal{L}$ and an arbitrary coherent sheaf $\mathcal{F}$. Prove:

   (a) There is a unique polynomial $P_{\mathcal{F}} \in \mathbb{Q}[T]$ such that $\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = P_{\mathcal{F}}(m)$ for all $m \in \mathbb{Z}$.

   (b) This polynomial can be written uniquely as $P_{\mathcal{F}}(T) = \sum'_n a_n \binom{T}{n}$ with $a_n \in \mathbb{Z}$.

   *(c) If $\mathcal{F} \neq 0$, the degree of $P_{\mathcal{F}}$ is equal to the dimension of the support of $\mathcal{F}$ and the highest coefficient of $P_{\mathcal{F}}$ is positive.

   (d) If X is a smooth connected curve and k is algebraically closed, the highest coefficient of $P_{\mathcal{O}_X}$ is $\deg(\mathcal{L})$.

   *(e) Repeat the same with an arbitrary invertible sheaf $\mathcal{L}$, assuming only in (c) that $\mathcal{L}$ is ample.

4. Let k be a field. Show that for any $f \in k((t))^\times$ and any $n \in \mathbb{Z}$ we have
   $$\mathrm{res}_t(f^n df) \;=\; \begin{cases} \mathrm{ord}_t(f) & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

5. Let k be an algebraically closed field of characteristic $\neq 2$. Let X be the connected smooth projective curve over k with the affine equation $y^2 = f(x)$ for a separable polynomial $f(x) \in k[x]$ of degree 3. Denote the function field of X by K.

   (a) Show that $\Gamma(X, \Omega_{X/k}) = k \cdot \frac{dx}{y}$.

   (b) Verify the residue theorem for the rational differentials dx, $\frac{dx}{x}$, $\frac{x\,dx}{y} \in \Omega_{K/k}$ by explicitly computing all residues.