Algebraic Geometry II

HS 2017

# Exercise Sheet 13

## Riemann-Roch, Embeddings in Projective Space

1. Let $k$ be an arbitrary field with algebraic closure $\bar{k}$. Let $X$ be a geometrically integral projective curve over $k$ with $h^1(X, \mathcal{O}_X) = 0$. Show:
   (a) The base change $X_{\overline{k}}$ is isomorphic to $\mathbb{P}^1_{\overline{k}}$.
   (b) The curve $X$ is isomorphic to a plane curve of degree 2.
   (c) We have $X \cong \mathbb{P}^1_k$ if and only if $X(k) \neq \varnothing$.

*2. Let $X$ be an irreducible smooth projective curve of genus $g = 1$ over an algebraically closed field $k$. Show that there exists a locally free sheaf of rank 2 on $X$ which is not a direct sum of invertible sheaves.

(For instance let $i_P$ denote the embedding of a closed point $P$ into $X$, let $\mathcal{E}$ be the kernel of a homomorphism $\mathcal{O}_X(P) \oplus \mathcal{O}_X \to i_{P*}k$ which is non-zero on each direct summand, and show that the resulting short exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}_X \xrightarrow{\binom{1}{0}} \mathcal{E} \xrightarrow{(0,1)} \mathcal{O}_X \longrightarrow 0$$

does not split.)

3. Let $X$ be a smooth, irreducible curve of genus $g$ over an algebraically closed field. Let $D$ be an effective divisor on $X$. Show that:
   (a) $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D + 1$.
   (b) $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D$ if and only if $\deg(D) \geqslant 1$ and $g \geqslant 1$.
   (c) $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D - 1$ if $\deg(D) \geqslant 2$ and $X$ is not hyperelliptic.

4. Let $X$ be a curve of genus 2. Show that a divisor $D$ on $X$ is very ample if and only if $\deg D \geqslant 5$.

5. Let $X$ be an irreducible smooth plane projective curve of degree 4 over an algebraically closed field $k$.
   (a) Show that the given embedding $X \hookrightarrow \mathbb{P}^2_k$ is the canonical embedding of $X$.
   (b) Deduce that $X$ is not hyperelliptic.
   (c) Show that the effective canonical divisors on $X$ are precisely the divisors of the form $X \cap L$ for all lines $L$ in $\mathbb{P}^2_k$.