Algebraic Geometry II

HS 2017

Exercise Sheet 14

Hyperelliptic Curves, Coverings

1. Let k be a perfect field of characteristic $p > 0$.

   (a) Let K be a finitely generated field extension of k of transcendence degree 1. Prove that for any $r \geqslant 1$ the only purely inseparable extension of degree p^r of K is the overfield $\{x^{1/p^r} \mid x \in K\}$.

   (b) Deduce that for every purely inseparable finite morphism of curves $X \to Y$ over k we have $g(X) = g(Y)$.

2. Let k be an algebraically closed field of characteristic 2 and let $g \geqslant 1$. Show that the smooth projective curve with the affine equation

$$y^2 + y = x^{2g+1}$$

   is hyperelliptic of genus g. Hence there exist hyperelliptic curves of every genus $\geqslant 1$ in characteristic 2.

3. Let $F \in k[x]$ be a separable polynomial of even degree $\geqslant 2$ over an algebraically closed field k with $\operatorname{char} k \neq 2$. Let X be the smooth projective curve over k with the affine equation $y^2 = F(x)$ and let $R := k[x,y]/(y^2 - F(x))$. Show that the following properties are equivalent:

   (a) There exist $A, B \in k[x]$ with $B \neq 0$ such that $A^2 - FB^2 = 1$.

   (b) $R^\times \neq k^\times$.

   (c) Let $P_1, P_2 \in X$ be the two points at infinity where x has a pole. Then the divisor class $[P_1 - P_2] \in \mathrm{Cl}^0(X)$ is an element of finite order.

   **Give examples where these properties hold and where they don't.

4. Let k be an algebraically closed field of characteristic $\neq 2$. An *elliptic curve* is an irreducible smooth projective curve of genus 1. Prove:

   (a) Show that for any two distinct closed points P and Q on an elliptic curve E there exists an automorphism $\sigma\colon E \to E$ of order 2 with $\sigma(P) = Q$.

   (b) For any $\lambda \in k \smallsetminus \{0,1\}$ the curve $E_\lambda \subset \mathbb{P}^2_k$ that is given by the equation

$$ZY^2 = X(X-Z)(X-\lambda Z)$$

   is an elliptic cuve.

(c) Show that any elliptic curve E is isomorphic to some such E_λ.

(d) Show that $E_\lambda \cong E_\mu$ if and only if

$$\mu \in \left\{\lambda, \tfrac{1}{\lambda}, 1-\lambda, \tfrac{1}{1-\lambda}, \tfrac{\lambda}{\lambda-1}, \tfrac{\lambda-1}{\lambda}\right\}.$$

(e) The *j-invariant* of an elliptic curve E is the element

$$j(E) := 2^8 \cdot \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} \in k$$

for any $\lambda \in k$ with $E_\lambda \cong E$. Show that $E \mapsto j(E)$ induces a bijection from the set of isomorphism classes of elliptic curves over k to k.

5. Show that the hyperelliptic curve over $\mathbb{C}$ with the affine equation $y^2 = x^5 - x$ has precisely 48 automorphisms.

6. Let k be an algebraically closed field of characteristic $p \geqslant 5$ and consider the hyperelliptic curve X of genus g given by

$$y^2 = x^p - x.$$

Show that $|\operatorname{Aut}(X)| \geqslant 2p(p^2-1) > 16g^3$.