D-MATH | Algebraic Geometry II | HS 2017
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Exercise Sheet 2

Coherent and Quasi-coherent Sheaves

Fix a locally noetherian scheme X.

1. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals that is locally free as an $\mathcal{O}_X$-module. Show that $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module, unless ... what?

*2. For any short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, show that if two of $\mathcal{F}'$, $\mathcal{F}$, $\mathcal{F}''$ are quasi-coherent, resp. coherent, so is the third.

3. Show that if $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, so is $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

4. Consider a morphism $f\colon X \to Y$. Is there a natural homomorphism between $(f_*\mathcal{E}) \otimes_{\mathcal{O}_Y} (f_*\mathcal{F})$ and $f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$? When is it an isomorphism?

5. Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{G}$ and $\mathcal{G}'$ be two $\mathcal{O}_Y$-modules. Define a natural homomorphism of $\mathcal{O}_X$-modules

$$\alpha\colon f^*\mathscr{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{G}') \to \mathscr{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, f^*\mathcal{G}'),$$

functorial in $\mathcal{G}$ and $\mathcal{G}'$. Show that α is an isomorphism if $\mathcal{G}$ is locally free of finite rank.

6. Prove that for any noetherian scheme X:

(a) Suppose X is affine. Prove that any quasi-coherent sheaf on X is a sum of coherent subsheaves.

(b) For general X, show that any coherent sheaf on an open subscheme of X is the restriction of a coherent sheaf on X.

**(c) Prove (a) for general X.

Hint: For (b), first prove the affine case. For the embedding $j : U \hookrightarrow X$ and a coherent sheaf $\mathcal{F}$ on U look at the sheaf $j_*\mathcal{F}$.