D-MATH | Algebraic Geometry II | HS 2017
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Exercise Sheet 3

Functoriality, Quasi-coherent Sheaves on $\mathrm{Proj}(R)$

1. (*Tensor Operations on Sheaves*) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and $n \in \mathbb{Z}^{\geqslant 0}$. We define the *n-th exterior power* $\bigwedge^n \mathcal{F}$ (resp. *n-th symmetric power* $\mathrm{Sym}^n \mathcal{F}$) of $\mathcal{F}$ by taking the sheaf associated to the presheaf which to each open set U assigns the $\mathcal{O}_X(U)$-module $\bigwedge^n_{\mathcal{O}_X(U)} \mathcal{F}(U)$ $\big($resp. $\mathrm{Sym}^n_{\mathcal{O}_X(U)} \mathcal{F}(U)\big)$.

   (a) Suppose that if $\mathcal{F}$ is locally free of rank r, then $\bigwedge^n \mathcal{F}$ and $\mathrm{Sym}^n \mathcal{F}$ are locally free of ranks $\binom{r}{n}$ and $\binom{r+n-1}{n}$ respectively.

   (b) With $\mathcal{F}$ as in (a) show that for each $n = 0, \ldots, r$ the multiplication map $\bigwedge^n \mathcal{F} \otimes \bigwedge^{r-n} \mathcal{F} \to \bigwedge^r \mathcal{F}$ is a perfect pairing, i.e., it induces an isomorphism $\bigwedge^n \mathcal{F} \cong (\bigwedge^{r-n} \mathcal{F})^\vee \otimes \bigwedge^r \mathcal{F}$.

   (c) Let $f\colon X \to Y$ be a morphism of schemes. Show that f^* commutes with $\bigwedge^n$ and Sym^n.

2. Let $f\colon X \to Y$ be a morphism of schemes. Recall that if $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{E}$ is an $\mathcal{O}_Y$-module, then there is a natural homomorphism $f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \to f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E})$. Show that this is an isomorphism when $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank.

*3. Let $f\colon X \to Y$ be a quasicompact, separated morphism of schemes, and suppose $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. Show that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

4. Let R be a ring. Consider a graded ideal $\mathfrak{a} \subset S := R[X_0, \ldots, X_n]$ and let $X := \mathrm{Proj}\, S/\mathfrak{a}$ and let $i\colon X \hookrightarrow \mathbb{P}^n_R$ be the associated closed embedding. Show that for any $k \in \mathbb{Z}$ there is a natural isomorphism $i^*\mathcal{O}_{\mathbb{P}^n_R}(k) \cong \mathcal{O}_X(k)$.

5. For graded modules over a graded ring R: Is the functor $M \mapsto \tilde{M}$ faithful? Full? Does it reflect isomorphisms?

6. Let $i\colon \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$ be the Segre embedding. Prove that $i^*\mathcal{O}(1) \cong \mathrm{pr}_1^*\mathcal{O}(1) \otimes \mathrm{pr}_2^*\mathcal{O}(1)$. What is $i^*\mathcal{O}(1)$ for the d-uple embedding $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$?