D-MATH
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Algebraic Geometry II

HS 2017

Exercise Sheet 4

Invertible Sheaves, Morphisms to Projective Space

1. Let $X = \operatorname{Proj} R$ for a graded ring R. Consider the graded ring $R' := \bigoplus_{d\geqslant 0} \mathcal{O}_X(d)(X)$. Is there a natural isomorphism $X \cong \operatorname{Proj} R'$?

2. Let $X = \operatorname{Proj} R$ for a graded ring R that is generated by finitely many elements of R_1 over a noetherian ring R_0. Prove that a sheaf of $\mathcal{O}_X$-modules is coherent if and only if it is isomorphic to $\tilde{M}$ for a finitely generated graded R-module M.

3. Let $\mathcal{L}$ and $\mathcal{L}'$ be invertible sheaves on a noetherian scheme X. Show:
   (a) If $\mathcal{L}$ is ample, there exists an $n_0 \in \mathbb{Z}$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is very ample for all $n \geqslant n_0$.
   (b) If $\mathcal{L}$ is ample and there exists an integer $n' > 0$ such that $\mathcal{L}'^{\otimes n'}$ is generated by its global sections, then $\mathcal{L} \otimes \mathcal{L}'$ is ample.
   (c) If $\mathcal{L}$ and $\mathcal{L}'$ are ample, then $\mathcal{L} \otimes \mathcal{L}'$ is ample.

4. Let $f\colon X \to Y$ be a finite morphism of noetherian schemes. Let $\mathcal{L}$ be an ample invertible sheaf on Y. Show that $f^*\mathcal{L}$ is ample.

5. Let X be the affine line over a field k with the origin doubled. Calculate $\operatorname{Pic} X$, determine which invertible sheaves are generated by global sections, and then show that there is no ample invertible sheaf on X.