Exercise Sheet 5

Projective Morphisms, Invertible Sheaves, Relative Proj

1. Prove that a morphism $f\colon X \to Y$ is projective if and only if it is quasiprojective and proper.

2. Let $f\colon X \to Y$ be a quasi-compact morphism. Let $\mathcal{L}$ and $\mathcal{L}'$ be invertible sheaves on X that are relatively ample over Y. Show that $\mathcal{L}\otimes\mathcal{L}'$ is relatively ample over Y.

3. Let $\mathcal{S} = \bigoplus_{d\geqslant 0} \mathcal{S}_d$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Then for each open affine $U \subset X$, we have the graded $\mathcal{O}_X(U)$-modules $\mathcal{S}(U) = \bigoplus_{d\geqslant 0} \mathcal{S}_d(U)$. Show that there exists a natural scheme $\pi\colon \operatorname{Proj}\mathcal{S} \to X$ together with isomorphisms $\eta_U\colon \pi^{-1}(U) \xrightarrow{\sim} \operatorname{Proj}\mathcal{S}(U)$ over U for all open affine subschemes $U \subset X$. Show that there is a natural invertible sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}\mathcal{S}$ whose restriction to each $\pi^{-1}(U)$ corresponds to the previously known sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}\mathcal{S}(U)$ via η_U. We call $\operatorname{Proj}\mathcal{S}$ the *relative* Proj of $\mathcal{S}$.

   Show that for any projective morphism $f\colon Y \to X$ with relatively very ample invertible sheaf $\mathcal{L}$ on Y there is a natural isomorphism $Y \cong \operatorname{Proj}\bigoplus_{d\geqslant 0} f_*\mathcal{L}^{\otimes d}$.

4. For any integer $n \geqslant 0$ consider the morphism

   $$X := \mathbb{A}^{n+1} \smallsetminus \{0\} \xrightarrow{\pi} Y := \mathbb{P}^n, \quad (x_0, \ldots, x_n) \mapsto (x_0 : \ldots : x_n).$$

   Prove that π is affine and determine the sheaf of $\mathcal{O}_X$-algebras $\pi_*\mathcal{O}_X$.

5. (a) Prove that a scheme X is affine if and only if there exist sections $s_1, \ldots, s_n \in \mathcal{O}_X(X)$ generating the unit ideal such that for each i the open subset $D_{s_i} = X \smallsetminus V(s_i)$ is affine.

   (b) Prove that for any morphism $f : X \to Y$, the following are equivalent:

      i. There exists an affine open covering $Y = \bigcup V_i$ such that each $f^{-1}(V_i)$ is affine.

      ii. For every open affine $V \subset Y$ the inverse image $f^{-1}(V)$ is affine.

Please turn over

*6. (*Vector bundles versus locally free sheaves*)

A *vector bundle of rank* $n \in \mathbb{Z}^{\geqslant 0}$ *over* X is a scheme V over X together with morphisms $+\colon V \times_X V \to V$ and $\cdot\colon \mathbb{A}^1 \times V \to V$ and $0\colon X \to V$ over V, such that there exists an open covering $X = \bigcup_\alpha U_\alpha$ and isomorphisms $f^{-1}(U_\alpha) \cong \mathbb{A}^n \times U_\alpha$ over U_α, such that the morphisms $+, \cdot, 0$ correspond to the morphisms

$$\begin{aligned} \mathbb{A}^n \times \mathbb{A}^n \times U_\alpha &\to \mathbb{A}^n \times U_\alpha, & ((x_1, \ldots, x_n), (y_1, \ldots, y_n), u) &\mapsto ((x_1 + y_1, \ldots, x_n + y_n), u) \\ \mathbb{A}^1 \times \mathbb{A}^n \times U_\alpha &\to \mathbb{A}^n \times U_\alpha, & (\lambda, (x_1, \ldots, x_n), u) &\mapsto ((\lambda x_1, \ldots, \lambda x_n), u) \\ U_\alpha &\to \mathbb{A}^n \times U_\alpha, & u &\mapsto ((0, \ldots, 0), u) \end{aligned}$$

Homomorphisms of vector bundles are morphisms of schemes over X which make certain commutative diagrams with the morphisms $+, \cdot, 0$ that one can guess. A vector bundle of rank 1 is called a *line bundle.*

(a) Look up the definition of vector bundles on a differentiable manifold and compare.

(b) Write down the commutative diagrams for morphisms of vector bundles.

(c) Why do the above conditions not include analogues of the usual vector space axioms? What would these analogues say?

(d) For any vector bundle V define the sheaf of sections $\mathcal{V}$ as a sheaf of $\mathcal{O}_X$-modules and extend this to a functor.

(e) Prove that this induces an equivalence from the category of vector bundles of all ranks to the full subcategory of quasi-coherent sheaves on X that are locally free of some rank.

(f) Discuss what is wrong with [Görtz and Wedhorn, Definition 11.5]. Construct a concrete example to show that (e) does not hold with that definition.

(g) Promise to never confuse vector bundles with locally free sheaves, or line bundles with invertible sheaves, even if many other people do.