Exercise Sheet 6

Picard Group, Divisors, Divisor Classes

Let k be a field.

1. Prove that $\mathrm{Aut}_k(\mathbb{P}^n_k)$ is naturally isomorphic to $\mathrm{PGL}_{n+1}(k)$.

2. Consider a morphism $\varphi\colon \mathbb{P}^n_k \to \mathbb{P}^m_k$ over k which does not factor through a k-valued point of $\mathbb{P}^m_k$. Show that φ can be obtained as the composition of (1) the d-uple embedding $\mathbb{P}^n_k \to \mathbb{P}^N_k$ for a uniquely determined $d \geqslant 1$, (2) a linear projection $\mathbb{P}^N_k - L \to \mathbb{P}^m_k$, where L is a linear subspace of $\mathbb{P}^N_k$, and (3) an automorphism of $\mathbb{P}^m_k$. Deduce that $\dim \varphi(\mathbb{P}^n_k) = n \leqslant m$ and that φ has finite fibers.

3. Consider the nodal cubic curve $X := V(C(C-B)A-B^3) \subset \mathbb{P}^2_k$. Prove that $\mathrm{Pic}(X) \cong k^\times \times \mathbb{Z}$.
(Hint: Recall that X has the normalization $\pi\colon \mathbb{P}^1_k \twoheadrightarrow X$. To describe an invertible sheaf $\mathcal{L}$ on X, describe $\pi^*\mathcal{L}$ and find out which additional information is necessary to determine $\mathcal{L}$.)

*4. Determine the Picard group of the cuspidal cubic curve $V(Y^2Z - X^3) \subset \mathbb{P}^2_k$.

5. (a) For any noetherian integral scheme X and any irreducible closed subscheme $Y \subset X$ of codimension 1, construct a natural exact sequence $\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \smallsetminus Y) \to 0$.
   (b) Prove that $\mathrm{Cl}(\mathbb{P}^n_k \smallsetminus V(f)) \cong \mathbb{Z}/d\mathbb{Z}$ for any irreducible homogeneous polynomial f of degree $d > 0$.

6. For any locally factorial noetherian separated integral scheme X and any $n \geqslant 0$ prove that there are natural isomorphisms
   (a) $\mathrm{Cl}(X \times \mathbb{A}^n) \cong \mathrm{Cl}(X)$ and
   (b) $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z}$.

7. Which of the following divisors is principal, resp. ample, resp very ample, resp. equivalent to an effective divisor?
   (a) $D = V(X^3+Y^3+Z^3) - V(X^2+Y^2+Z^2)$ on $\mathbb{P}^2_k$.
   (b) $D = -P$ for $P := V(X-1, Y)$ on $X := \mathrm{Spec}\,\mathbb{R}[X,Y]/(X^2+Y^2-1)$.
   (c) $D = a\,\mathrm{diag}\,\mathbb{P}^1_k + b\,\mathrm{pr}_1^*\,P + c\,\mathrm{pr}_2^*\,P$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ for $P \in \mathbb{P}^1(k)$ and $a, b, c \in \mathbb{Z}$.