D-MATH | Algebraic Geometry II | HS 2017
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Exercise Sheet 7

Divisors, Divisor Classes, Differentials

1. Consider the nonsingular cubic curve $X := V(Y^2Z - X^3 - 4XZ^2) \subset \mathbb{P}^2_{\mathbb{C}}$ and its closed points $P_0 := (0:1:0)$ and $P := (2:4:1)$ and $Q := (0:0:1)$. Which of the following divisors on X is principal, resp. equivalent to an effective divisor?
   (a) $D_1 := P - P_0$.
   (b) $D_2 := P + Q - P_0$.
   (c) $D_3 := 2P + Q - 3P_0$.

2. Let X be as in the previous exercise. Recall that its set of closed points $|X|$ possesses a natural abelian group structure with zero element P_0.
   (a) Show that $P \in |X| \setminus \{P_0\}$ has order 2 in $|X|$ if and only if the tangent line at P passes through P_0.
   (b) An *inflection point* of a plane curve is a nonsingular point P of the curve, whose tangent line has intersection multiplicity $\geqslant 3$ with the curve at P. Show that the inflection points in $|X|$ are precisely P_0 and all points of order 3.

**3. Let $X = \operatorname{Spec} k[S,T,U]/(UT - S^2)$, and let C be the closed subscheme $V(S,T)$ of X.
   (a) Show that $C \cong \mathbb{A}^1_k$ is a prime Weil divisor on X and that $\mathrm{Cl}(X)$ is a group of order 2 which is generated by the class of C.
   (b) Show that $\mathrm{DivCl}(X) = 0$ and deduce that $\mathrm{cyc}\colon \mathrm{Div}(X) \to Z^1(X)$ is not surjective.

4. Consider ring homomorphisms $A \to B \to C$. Prove the following statements from the course:
   (a) For any multiplicative system $S \subset B$, there is a natural isomorphism of $S^{-1}B$-modules $S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A}$.
   (b) There is a natural exact sequence
$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$
   (c) If $C = B/J$ for an ideal $J \subset B$, there is a natural exact sequence
$$J/J^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0.$$

5. Compute $\Omega_{B/A}$ in the following cases, and determine the largest open subscheme of $\operatorname{Spec} B$ over which $\Omega_{B/A}$ is locally free.

   (a) $B = A[X,Y]/(XY - t)$ for a discrete valuation ring A with uniformizer t.

   (b) $B = k[X,Y,Z,W]/(XY - ZW)$ where $A = k$ is a field.

   (c) $B = k[X]/(X^n)$ for a field $A = k$ and an integer $n \geqslant 1$.

6. Let k be perfect of characteristic $p > 0$. Let K be an extension of k. Show that $k \subset K^p$ and that the canonical homomorphism $\Omega^1_{K/k} \to \Omega^1_{K/K^p}$ is an isomorphism.

7. Construct an inseparable field extension L/K with $\Omega_{L/K} = 0$.