D-MATH
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Algebraic Geometry II

HS 2017

Exercise Sheet 8

Sheaves of Differentials, Canonical Sheaf, Smoothness

Let k be a field. Recall that a *variety* over k is a reduced scheme X of finite type over k. We say that X is *nonsingular* if it is regular at every point.

1. Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on X. Show that any point $x \in X$, such that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, possesses an open neighborhood $U \subset X$ such that $\mathcal{F}|_U$ is free. Deduce that $\mathcal{F}$ is locally free if and only if $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all closed points $x \in X$.

2. (a) Let Y_1 and Y_2 be schemes over X and let $Y := Y_1 \times_X Y_2$. Construct a natural isomorphism
$$\Omega_{Y/S} \cong \mathrm{pr}_1^*\Omega_{Y_1/X} \oplus \mathrm{pr}_2^*\Omega_{Y_2/X}.$$
(b) If Y_1 and Y_2 are nonsingular varieties over a perfect field k, construct a natural isomorphism
$$\omega_{Y/k} \cong \mathrm{pr}_1^*\omega_{Y_1/k} \otimes \mathrm{pr}_2^*\omega_{Y_2/k}.$$

3. Let X be a nonsingular variety over an algebraically closed field k. We call $\mathcal{T}_X := \mathscr{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ the *(relative) tangent sheaf of X (over k).* A global section of $\mathcal{T}_X$ is called a *tangent field on X.*

   (a) Show that $\mathcal{T}_X$ is locally free. What is its rank?

   (b) Describe $\mathcal{T}_{\mathbb{P}^n_k}$ by an explicit short exact sequence.

   (c) Does $\mathbb{P}^1_k$ possess a nowhere vanishing tangent field?

   **(d) Does $\mathbb{P}^n_k$ possess a nowhere vanishing tangent field for arbitrary n?

*4. Let $i\colon Y \hookrightarrow X$ be a closed immersion of codimension 1 of a nonsingular variety X over an algebraically closed field k, whose ideal sheaf $\mathcal{J}$ can be locally generated by one element at every point. We define the *canonical sheaf* of such Y as
$$\omega^\circ_{Y/k} := i^*\omega_{X/k} \otimes i^*(\mathcal{J}/\mathcal{J}^2)^\vee.$$

   (a) Prove that $\omega^\circ_{Y/k}$ is an invertible sheaf.

   (b) Verify that $\omega^\circ_{Y/k} \cong \omega_{Y/k}$ if Y is nonsingular.

   (c) Determine $\Omega_{Y/k}$ and $\omega^\circ_{Y/k}$ for the nodal curve $Y = V(C(C-B)A - B^3) \subset \mathbb{P}^2_k$ and explain the difference.

5. (*Smooth Morphisms and the Jacobian Criterion*) Let $f\colon X \to Y$ be a morphism of schemes and $d \geqslant 0$ an integer. We say that f is *smooth of relative dimension* d at $x \in X$, if there exist affine open neighborhoods U of x and $V = \operatorname{Spec} R$ of $f(x)$ such that $f(U) \subset V$, and an open immersion

$$j\colon U \hookrightarrow \operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d})$$

of R-schemes for suitable n and f_i, such that the Jacobian matrix

$$J_{f_1,\dots,f_{n-d}}(x) := \left(\frac{\partial f_i}{\partial T_j(x)}\right)_{i,j} \in M_{n-d\times n}(\kappa(x))$$

has rank $n-d$. We call f *smooth* if it is smooth at all points $x \in X$. Show:

(a) Smoothness is local on the source and the target.

(b) Smoothness is invariant under base change.

(c) Smoothness is invariant under composition.

(d) Every open immersion is smooth of relative dimension 0.

(e) The set of points of X at which f is smooth is open.

6. Let X be a scheme of finite type over k, where k is perfect. We call X *smooth* over k if the structure morphism $X \to \operatorname{Spec} k$ is smooth. Assume that X is irreducible, and show that X is smooth over k if and only if $\Omega^1_{X/k}$ is locally free of rank $\dim X$.