Algebraic Geometry II

HS 2017

Exercise Sheet 9

Čech Cohomology

1. Show that the complex of abelian groups $\ldots \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \ldots$ is acyclic but not contractible.

2. Let X be a separated and quasicompact scheme and let $(\mathcal{F}_i)_{i\in I}$ be a filtered direct system of quasicoherent sheaves on X. Show that for any $p \geqslant 0$ there is a natural isomorphism
$$\varinjlim_i H^p(X, \mathcal{F}_i) \cong H^p(X, \varinjlim_i \mathcal{F}_i).$$

3. Let X be a scheme.

   (a) Construct a natural isomorphism $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$.

   *(b) Suppose that X is integral and let $\mathcal{K}_X$ denote the constant sheaf with values in $K(X)$. Show that the exact sequence
$$1 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 1$$
   induces the isomorphism $\operatorname{DivCl}(X) \xrightarrow{\sim} \operatorname{Pic}(X)$ from §5.9 of the course.

4. Compute $H^*(X, \mathcal{O}_X)$ where $X = \mathbb{P}^2_k \smallsetminus \{(0:0:1)\}$ and for $X = \mathbb{A}^2_k \smallsetminus \{(0,0)\}$ for a field k. Conclude that X is not affine.

*5. Find a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on an affine scheme X for which $H^n(X, \mathcal{F}) \neq 0$ for some $n > 0$.