D-MATH | Algebraic Geometry II | HS 2017

Prof. Richard Pink

Solutions 1

$\mathcal{O}_X$-Modules, Quasi-coherent Sheaves

Convention: Given a morphism of sheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ on a space X and an open set $U \subset X$, we denote the U-component of α by $\alpha_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$.

Fix a scheme $(X, \mathcal{O}_X)$.

1. Explain in all details why and how the tensor product of sheaves of $\mathcal{O}_X$-modules is functorial in both variables.

   Solution: We first consider the association $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. To show that this defines a functor from the category of $\mathcal{O}_X$-modules to itself, we must show that for every morphism $\varphi\colon \mathcal{F} \to \mathcal{F}'$ we obtain a morphism "$\varphi \otimes \mathrm{id}_{\mathcal{G}}$" from $\mathcal{F} \otimes \mathcal{G} \to \mathcal{F}' \otimes \mathcal{G}$ such that:

   (a) $\mathrm{id}_{\mathcal{F}} \otimes \mathrm{id}_{\mathcal{G}} = \mathrm{id}_{\mathcal{F}\otimes\mathcal{G}}$ and

   (b) $(\psi \circ \varphi) \otimes \mathrm{id}_{\mathcal{G}} = (\psi \otimes \mathrm{id}_{\mathcal{G}}) \circ (\varphi \otimes \mathrm{id}_{\mathcal{G}})$.

   Let $\varphi\colon \mathcal{F} \to \mathcal{F}'$ be a morphism. For each $U \subset X$ open we obtain a composition of morphisms

   $$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \to \mathcal{F}'(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \to (\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{G})(U),$$

   where the first morphism is given by $s \otimes t \mapsto \varphi_U(s) \otimes t$, and the second by the sheafification. This is compatible with restriction and we obtain a morphisms of presheaves $(\mathcal{F} \otimes \mathcal{G})_{pre} \to (\mathcal{F}' \otimes \mathcal{G})_{pre} \to \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{G}$. We denote the first arrow by $(\varphi \otimes \mathrm{id}_{\mathcal{G}})_{pre}$ and define $\varphi \otimes \mathrm{id}_{\mathcal{G}}$ to be the morphism of sheaves induced by the compositon via the universal property of the sheafification. To check (a), it suffices to observe that we have a commutative diagram:

   $$\begin{array}{ccc} (\mathcal{F} \otimes \mathcal{G})_{pre} & \xrightarrow{(\mathrm{id}_{\mathcal{F}}\otimes \mathrm{id}_{\mathcal{G}})_{pre}} & (\mathcal{F} \otimes \mathcal{G})_{pre} \\ \downarrow & & \downarrow \\ \mathcal{F} \otimes \mathcal{G} & \xrightarrow{\mathrm{id}_{\mathcal{F}\otimes\mathcal{G}}} & \mathcal{F} \otimes \mathcal{G}, \end{array}$$

   where the vertical arrows are the sheafification morphisms. By the universal property, the bottom arrow is unique. Since this was defined to be $\mathrm{id}_{\mathcal{F}} \otimes \mathrm{id}_{\mathcal{G}}$, it follows that $\mathrm{id}_{\mathcal{F}} \otimes \mathrm{id}_{\mathcal{G}} = \mathrm{id}_{\mathcal{F}\otimes\mathcal{G}}$

For (b), let $\varphi\colon \mathcal{F} \to \mathcal{F}'$ and $\psi\colon \mathcal{F}' \to \mathcal{F}''$ be morphisms of $\mathcal{O}_X$-modules. Again we have a commutative diagram:

$$\begin{array}{ccccc}
(\mathcal{F} \otimes \mathcal{G})_{pre} & \xrightarrow{(\varphi \otimes \mathrm{id}_{\mathcal{G}})_{pre}} & (\mathcal{F}' \otimes \mathcal{G})_{pre} & \xrightarrow{(\psi \otimes \mathrm{id}_{\mathcal{G}})_{pre}} & (\mathcal{F}'' \otimes \mathcal{G})_{pre} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F} \otimes \mathcal{G} & \xrightarrow{\varphi \otimes \mathrm{id}_{\mathcal{G}}} & \mathcal{F} \otimes \mathcal{G} & \xrightarrow{\psi \otimes \mathrm{id}_{\mathcal{G}}} & \mathcal{F} \otimes \mathcal{G},
\end{array}$$

Since the composition of the top arrows is equal to $((\psi \circ \varphi) \otimes \mathrm{id}_{\mathcal{G}})_{pre}$, it again follows from the uniqueness in the universal property that $(\psi \otimes \mathrm{id}_{\mathcal{G}}) \circ (\varphi \otimes \mathrm{id}_{\mathcal{G}}) = (\psi \circ \varphi) \otimes \mathrm{id}_{\mathcal{G}}$.

The functoriality of $\mathcal{F} \otimes \mathcal{G}$ in the second variable is completely analogous.

2. (*Basic properties of the sheaf of homomorphisms*) Consider $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{F}_i$. In each of the following cases construct a natural isomomorphism, or a natural homomorphism and discuss under which additional conditions this is an isomorphism.

   (a) Between $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ and $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ for any $x \in X$.

   (b) Between $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$ and $\mathcal{F}$.

   (c) Between $\mathscr{H}om_{\mathcal{O}_X}(\bigoplus_{i \in I} \mathcal{F}_i, \mathcal{G})$ and $\prod_{i \in I} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}_i, \mathcal{G})$.

   (d) Between $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \prod_{i \in I} \mathcal{F}_i)$ and $\prod_{i \in I} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}_i)$.

   (e) Between $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H})$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$.

   (f) Between $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H})$ and $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$.

Solution:

(a) The natural homomorphism $\alpha\colon \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ is constructed as follows: Let $x \in X$ and consider an element $[(\varphi, U)] \in \mathscr{H}om_x(\mathcal{F}, \mathcal{G})_x$ where $U \subset X$ is an open neighborhood of x and $\varphi \in \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$. We define $\alpha([(\varphi, U)]) := \varphi_x$, where $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ denotes the morphism induced on stalks by φ.

This is not in general an isomorphism. For a counterexample consider $X := \operatorname{Spec} \mathbb{Z}$ and $\eta := (0) \in X$. Let $\mathcal{F} := \underline{\mathbb{Q}}$ be the constant sheaf and let $\mathcal{G} := \mathcal{O}_X$. Every open subset of X is isomorphic to $\operatorname{Spec} \mathbb{Z}[n^{-1}]$ for some $n \in \mathbb{Z}$. For each non-empty such $U \subset X$, the morphisms $\mathcal{O}_X(U) \hookrightarrow \mathrm{Quot}(\mathcal{O}_X(U)) = \mathbb{Q}$ endow $\mathcal{F}$ with the structure of an $\mathcal{O}_X$-module. We have $\mathrm{Hom}_{\mathcal{O}_{X,\eta}}(\mathcal{F}_\eta, \mathcal{G}_\eta) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$.

On the other hand, let $U = \operatorname{Spec} \mathbb{Z}[n^{-1}] \subset X$ be an arbitrary open subset. Then any $\varphi \in \mathrm{Hom}_{\mathcal{O}_U}(\underline{\mathbb{Q}}, \mathcal{O}_U)$ induces an $\mathbb{Z}[n^{-1}]$-linear morphism on global sections $\varphi_U\colon \mathbb{Q} \to \mathbb{Z}[n^{-1}]$. Such a morphism must be zero. Hence $\varphi = 0$, and we deduce that $\mathrm{Hom}_{\mathcal{O}_U}(\underline{\mathbb{Q}}, \mathcal{O}_U) = 0$. We thus have

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_\eta = \varinjlim_{\eta \in U \subset X} \mathrm{Hom}_{\mathcal{O}_U}(\underline{\mathbb{Q}}, \mathcal{O}_U) = 0.$$

Thus α cannot be an isomorphism.

One can show that α is an isomorphism in the case where $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation [Görtz and Wedhorn, Proposition 7.27]. The main point is that the functors $\mathcal{F} \mapsto \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x$ and $\mathcal{F} \mapsto \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ are left exact and commute with finite direct sums. The former functor is left exact by part (b) of exercise 3 and the fact that a sequence of sheaves is exact if and only if it is exact at the stalks. The latter is the composition of the exact functor $\mathcal{F} \mapsto \mathcal{F}_x$ and the left exact functor $M \mapsto \operatorname{Hom}_{\mathcal{O}_{X,x}}(M, \mathcal{G}_x)$, hence also left exact. All of these operations commute with finite direct sums: For the sheaf of homomorphisms this is part (c) of exercise 2, and for taking stalks this follows from the fact that stalks are colimits. Thus both functors commute with finite direct sums.

(b) Let $U \subset X$ be open. Consider the morphism

$$\alpha_U \colon \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{G}|_U) \to \mathcal{G}(U),\ \varphi \mapsto \varphi_U(1).$$

For any $s \in \mathcal{G}(U)$, there is a unique homomorphism $\varphi\colon \mathcal{O}_U \to \mathcal{G}|_U$ such that $\varphi_U(1) = s$. It follows that α_U is an isomorphism of $\mathcal{O}_X(U)$-modules. Since this construction is compatible with restriction, the α_U determine an isomorphism of sheaves $\alpha\colon \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \xrightarrow{\sim} \mathcal{G}$.

(c) and (d) There are natural isomorphisms

$$\prod_{i\in I} \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}_i|_U, \mathcal{G}|_U) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_U}(\bigoplus_{i\in I} \mathcal{F}_i|_U, \mathcal{G}|_U) \cong \operatorname{Hom}_{\mathcal{O}_U}((\bigoplus_{i\in I} \mathcal{F}_i)|_U, \mathcal{G}|_U),$$
$$\prod_{i\in I} \operatorname{Hom}_{\mathcal{O}_u}(\mathcal{G}|_U, \mathcal{F}_i|_U) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \prod_{i\in I} \mathcal{F}_i|_U) \cong \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, (\prod_{i\in I} \mathcal{F}_i)|_U),$$

for all $U \subset X$ open. The isomorphisms on the left were given in class. The isormorphisms on the right are induced by the canonical isomorphisms $\bigoplus_{i\in I} \mathcal{F}_i|_U \cong (\bigoplus_{i\in I} \mathcal{F}_i)|_U$ and $\prod_{i\in I} \mathcal{F}_i|_U \cong (\prod_{i\in I} \mathcal{F}_i)|_U$. These define isomorphisms on the corresponding sheaves of homomorphisms as in part (b).

(e) The following proof is very similar to the proof of the analogous statement for modules over a ring. Consider a morphism $\varphi\colon \mathcal{F} \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})$. Let $U \subset X$ be open. We obtain a morphism

$$\varphi_U \colon \mathcal{F}(U) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}|_U, \mathcal{H}|_U),\ s \mapsto \varphi_{U,s}.$$

Now define a morphism $\psi_U^\varphi\colon \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \to \mathcal{H}(U)$ of $\mathcal{O}_X(U)$-modules via $\psi_U^\varphi(s \otimes t) := (\varphi_{U,s})_U(t)$. By varying U and observing that this construction is compatible with restriction, we obtain a morphism from the presheaf $U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U)$ to $\mathcal{H}$. By the universal property of the sheafification, this factors through a unique morphism $\psi^\varphi\colon \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}$. The map $\varphi \mapsto \psi^\varphi$ yields a morphism

$$\alpha\colon \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}).$$

We now define an inverse. Let $\psi\colon \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}$ be a morphism. Let $U \subset X$ be open and consider $s \in \mathcal{F}(U)$. For each $V \subset U$ open, we have a morphism

$$\mathcal{G}(V) \to \mathcal{H}(V),\ t \mapsto \psi_V(\widetilde{s|_V \otimes t}),$$

where $\widetilde{s|_V \otimes t}$ denotes the image of $s|_V \otimes t$ under the sheafification morphism $\mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{G}(V) \to (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(V)$. Varying V we obtain a morphism of sheaves $\mathcal{G}|_U \to \mathcal{H}|_U$. We have thus constructed a morphism

$$\varphi_U^\psi\colon \mathcal{F}(U) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})(U).$$

This construction is compatible with restriction and yields a morphism φ^ψ of the corresponding sheaves. The map $\psi \mapsto \varphi^\psi$ is the desired inverse, and α is an isomorphism.

(f) The isomorphism from (e) yields natural isomorphisms

$$\alpha_U\colon \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))(U) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H})(U)$$

for all open $U \subset X$, and thus an isomorphism of sheaves

$$\alpha\colon \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}).$$

3. (*Exactness properties of* Hom *and* $\mathscr{H}om$) Prove the following:

   (a) A sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ of $\mathcal{O}_X$-modules is exact if and only if for all open subsets $U \subset X$ and for all $\mathcal{O}_U$-modules $\mathcal{G}$ the sequence

   $$0 \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}'|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}''|_U)$$

   of $\mathcal{O}_X(U)$-modules is exact if and only if for all $\mathcal{O}_X$-modules $\mathcal{G}$ the sequence

   $$0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}') \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}'')$$

   of $\mathcal{O}_X$-modules is exact.

   (b) A sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of $\mathcal{O}_X$-modules is exact if and only if for all open subsets $U \subset X$ and for all $\mathcal{O}_U$-modules $\mathcal{G}$ the sequence

   $$0 \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}''|_U, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}'|_U, \mathcal{G})$$

   of $\mathcal{O}_X(U)$-modules is exact if and only if for all $\mathcal{O}_X$-modules $\mathcal{G}$ the sequence

   $$0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathscr{H}om_{\mathcal{O}}(\mathcal{F}', \mathcal{G})$$

   of $\mathcal{O}_X$-modules is exact.

Solution:

(a) We want to show that the following statements are equivalent:

(i.) The sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \tag{1}$$
of $\mathcal{O}_X$-modules is exact.

(ii.) For all $\mathcal{O}_U$-modules $\mathcal{G}$, the sequence
$$0 \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}'|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}|_U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}''|_U) \tag{2}$$
of $\mathcal{O}_X(U)$-modules is exact.

(iii.) For all $\mathcal{O}_X$-modules $\mathcal{G}$, the sequence
$$0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}') \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}'') \tag{3}$$
of $\mathcal{O}_X$-modules is exact.

Since exactness can be tested on stalks, it follows that for all $U \subset X$ open the sequence $0 \to \mathcal{F}'|_U \to \mathcal{F}|_U \to \mathcal{F}''|_U$ is also exact. For the equivalence (i.)$\Leftrightarrow$(ii.), we may thus replace U by X in (ii.). For (i.)$\Rightarrow$(ii.), it suffices to show that the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, -)$ for any $\mathcal{O}_X$-module is left exact. This is done exactly as in the case of modules. For (ii.)$\Rightarrow$(i.), one also emulates the proof of the corresponding fact for modules. Alternatively, we can admit the fact that the category of $\mathcal{O}_X$-modules over a scheme X is abelian. In such a category $\mathcal{C}$, an exact sequence $0 \to A \to B \to C$ is exact iff $0 \to \mathrm{Hom}_{\mathcal{C}}(D, A) \to \mathrm{Hom}_{\mathcal{C}}(D, B) \to \mathrm{Hom}_{\mathcal{C}}(D, C)$ for all $D \in \mathcal{C}$ is exact.

Suppose (ii.) holds. Then taking sections over any open $U \subset X$ in (**??**) yields an exact sequence of the form (**??**). Since a sequence of sheaves $0 \to \mathcal{H}' \to \mathcal{H} \to \mathcal{H}''$ is exact if and only if $0 \to \mathcal{H}'(U) \to \mathcal{H}(U) \to \mathcal{H}(U)''$ is exact for all $U \subset X$ open, we conclude that (**??**) is exact. Thus (ii.)$\Rightarrow$(iii.).

For (iii.)$\Rightarrow$(ii.), let $U \subset X$ be open and suppose $\mathcal{G}$ is an $\mathcal{O}_U$-module. Let $j\colon U \hookrightarrow X$ denote the inclusion and consider the $\mathcal{O}_X$-module $j_!\mathcal{G}$ obtained by extending by zero. The sequence (**??**) for $j_!\mathcal{G}$ is exact by assumption. Since $j_!\mathcal{G}|_U = \mathcal{G}$ we can take sections over U to obtain (**??**). Taking sections is left exact, so it follows that (**??**) is also exact.

The proof of (b) is entirely analogous.

Note: By part (e) of exercise 2, the functor $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{G}, -)$ is a right adjoint and therefore left exact. Also, one gets the implication (iii.)$\Rightarrow$(i) directly by taking $\mathcal{G} := \mathcal{O}_X$ and applying part (b) of exercise 2.

4. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank.

(a) Show that $(\mathcal{E}^\vee)^\vee \cong \mathcal{E}$.

(b) For any $\mathcal{O}_X$-module $\mathcal{F}$, show that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes \mathcal{F}$.

(c) Let n be the rank of $\mathcal{E}$ and suppose that $\mathcal{F}$ is locally free of finite rank m. Show that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is locally free of rank nm.

Solution:

(a) We define a natural morphism $\mathcal{E} \to (\mathcal{E}^\vee)^\vee$ in analogy to the evaluation map for modules over a ring. Let $U \subset X$ be open. Consider the morphism

$$\alpha_U \colon \mathcal{E}(U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{E}^\vee|_U, \mathcal{O}_U),\ s \mapsto \alpha_U(s),$$

where $\alpha_U(s)$ is given on each open $V \subset U$ and $f \in \mathcal{E}^\vee(V)$ by $(\alpha_U(s))_V(f) = f_V(s|_V)$. The α_U commute with restriction, and we obtain a morphism $\alpha\colon \mathcal{E} \to (\mathcal{E}^\vee)^\vee$.

Since $\mathcal{E}$ is locally free of finite rank, it is of finite presentation. We can thus apply the result in the solution for part (a) of exercise 2 to obtain a natural isomorphism for any $x \in X$:

$$((\mathcal{E}^\vee)^\vee)_x = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}^\vee, \mathcal{O}_X)_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}((\mathcal{E}_x)^\vee, \mathcal{O}_{X,x}),$$

where $(-)^\vee$ on the right hand side denotes the dual as an $\mathcal{O}_{X,x}$-module. Then α induces the evaluation map on stalks $\mathcal{E}_x \to ((\mathcal{E}_x)^\vee)^\vee$. Since $\mathcal{E}$ is locally free of finite rank, the stalk $\mathcal{E}_x$ is a free $\mathcal{O}_{X,x}$-module of finite rank. Let A be a ring and M a free A-module of finite rank, and let $M^\vee := \mathrm{Hom}_A(M, A)$. It is a general fact that the evaluation map $M \to (M^\vee)^\vee$ is an isomorphism. The proof is the same as for vector spaces. It follows that α induces an isomorphism on stalks and is thus itself an isomorphism.

Note: After reducing to the case where $\mathcal{E}$ is free, one can show that α is an isomorphism directly by emulating the proof for modules. One can also use the fact that X is a scheme. Taking a small enough open affine $\mathrm{Spec}(A) \subset X$, we may assume $\mathcal{E} \cong (A^n)^\sim$. Using part (a) of exercise 5, we may commute $(-)^\vee$ and $(-)^\sim$, and we again reduce to the analogous statement for modules.

(b) See [Görtz and Wedhorn, Section 7.5].

(c) Since the question is local on the base, we may suppose that $\mathcal{E} = \mathcal{O}_X^n$ and $\mathcal{F} = \mathcal{O}_X^m$. We have a chain of isomorphisms,

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{O}_X^m) \cong \bigoplus_{n,m} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X^{nm}.$$

The first isomorphism results from (c) and (d) in exercise 2, where we've replaced all products with direct sums since the index set is finite. The second isomorphism is part (b) of exercise 2.

5. (a) If $X = \operatorname{Spec} A$, show that for A-modules M and N, **where M is finitely presented**, there is an isomorphism $\operatorname{Hom}_A(M,N)^{\sim} \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\tilde{M},\tilde{N})$ of $\mathcal{O}_X$-modules which is functorial in M and N.

   (b) Show that for quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, **where $\mathcal{F}$ is finitely presented**, the $\mathcal{O}_X$-module $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is again quasi-coherent.

Solution: *Note*: The initial version of this exercise omitted the assumption about finite presentation.

(a) We use the fact that if M is finitely presented and $S \subset A$ is multiplicatively closed, then there is a natural isomorphism

$$S^{-1}\operatorname{Hom}_A(M,N) \cong \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

See [Eisenbud, *Commutative Algebra*, Prop. 2.10]. For each standard affine open $\operatorname{Spec} A_f \cong D_f \subset X$, we have

$$\operatorname{Hom}_A(M,N)^{\sim}(D_f) = \operatorname{Hom}_A(M,N)_f \cong \operatorname{Hom}_{A_f}(M_f, N_f).$$

On the other hand, there are natural isomorphisms

$$\begin{aligned} \mathscr{H}om_{\mathcal{O}_X}(\tilde{M},\tilde{N})(D_f) &= \operatorname{Hom}_{\mathcal{O}_{D_f}}(\tilde{M}|_{D_f}, \tilde{N}|_{D_f}) \cong \\ \operatorname{Hom}_{\mathcal{O}_{D_f}}(\tilde{M_f}, \tilde{N_f}) &\cong \operatorname{Hom}_{A_f}(M_f, N_f). \end{aligned}$$

These isomorphisms were given in the lecture. It follows that there is a natural isomorphism $\operatorname{Hom}_A(M,N)^{\sim}(D_f) \cong \mathscr{H}om_{\mathcal{O}_X}(\tilde{M},\tilde{N})(D_f)$. This is compatible with restriction to other standard affine open subsets and thus defines the desired isomorphism of sheaves.

(b) For each affine open $U = \operatorname{Spec} A \subset X$, we wish to show that $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})|_U \cong \tilde{P}$ for some A-module P. Since $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent and $\mathcal{F}$ is finitely presented, we may suppose that $\mathcal{F} = \tilde{M}$ and $\mathcal{G} = \tilde{N}$ for A-modules M and N, with M finitely presented. Then the desired result follows directly from part (a), with $P = \operatorname{Hom}_A(M,N)$.

**6. Give an example of a scheme X and a set I such that $\mathcal{O}_X^I$ is not free.

Solution: See exercise 6 on *https://wiki.epfl.ch/ringsandmodules/documents/Rings and Modules sheet 3.pdf* (cut and paste the url into your web browser). For the solution, see *https://wiki.epfl.ch/ringsandmodules/documents/3.6.pdf.*