Solutions 10

Cohomology, Ampleness Criterion, Higher Direct Image

1. Let X be a projective scheme over a noetherian ring A. Consider a finite exact sequence $\mathcal{F}^1 \to \mathcal{F}^2 \to \ldots \to \mathcal{F}^r$ of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geqslant n_0$, the sequence of global sections

$$\mathcal{F}^1(n)(X) \to \mathcal{F}^2(n)(X) \to \ldots \to \mathcal{F}^r(n)(X)$$

is exact.

Solution: For each i, let $\varphi_i \colon \mathcal{F}^i \to \mathcal{F}^{i+1}$ denote the morphism in the exact sequence from the statement of the problem. Then for each $i \in \{1, \ldots, r-1\}$, we have a short exact sequences

$$0 \to \ker \varphi_{i+1} \to \mathcal{F}^{i+1} \to \operatorname{coker} \varphi_i \to 0,$$

and the $\ker \varphi_{i+1}$ and $\operatorname{coker} \varphi_i$ are also coherent sheaves on X.

The sheaf $\mathcal{O}_X(1)$ is very ample and hence ample. Fix $i \in \{1, \ldots r-1\}$. By the cohomological criterion for ampleness, there exists an $n_0 \in \mathbb{Z}$ such that for all $n \geqslant n_0$, we have $H^1(X, \ker \varphi_{i+1}(n)) = 0$. Since there are only finitely many i, we may choose an n_0 that works for all of them. The long exact sequences on cohomology thus yield short exact sequences

$$0 \to \ker \varphi_{i+1}(n)(X) \to \mathcal{F}^{i+1}(n)(X) \to \operatorname{coker} \varphi_i(n)(X) \to 0 \qquad (1)$$

for every i. These splice together to the sequence

$$\mathcal{F}^1(n)(X) \to \mathcal{F}^2(n)(X) \to \ldots \to \mathcal{F}^r(n)(X)$$

from the statement of the exercise, which is exact since the sequences in (1) are.

2. Consider separated quasicompact morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and a quasicoherent sheaf $\mathcal{F}$ on X. Prove that there are natural isomorphisms

$$R^p(g \circ f)_*\mathcal{F} \;\cong\; \begin{cases} R^p g_*(f_*\mathcal{F}) & \text{if } f \text{ is affine,} \\ g_*(R^p f_*\mathcal{F}) & \text{if } g \text{ is affine.} \end{cases}$$

Solution: We begin with the following lemma:

Lemma. *Let $f\colon X \to Y$ be an affine morphism with Y separated, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for every $p \in \mathbb{Z}$, the canonical homomorphism $H^p(X, \mathcal{F}) \to H^p(Y, f_*\mathcal{F})$ is an isomorphism.*

Proof. First note that X is separated over Y and hence itself separated, since finite morphisms are separated. Let $\mathcal{V} = (V_i)_{i \in I}$ be an affine cover of Y. Then $\mathcal{U} := \big(U_i := f^{-1}(V_i)\big)_{i \in I}$ is an affine cover of X such that $C^\bullet(\mathcal{V}, f_*\mathcal{F}) = C^\bullet(\mathcal{U}, \mathcal{F})$. We thus have

$$H^p(\mathcal{V}, f_*\mathcal{F}) = H^p(\mathcal{U}, \mathcal{F}).$$

Since X and Y are separated, these are isomorphic to the Čech cohomology groups and the desired result follows. □

Let $U \subset Z$ be an affine open subscheme, and let $V := (g \circ f)^{-1}(U)$. Suppose f is affine. A result from the course yields $R^p g_*(f_*\mathcal{F})(U) = H^p(g^{-1}(U), f_*\mathcal{F}|_{g^{-1}(U)})$. By the lemma, the latter is isomorphic to $H^p(V, \mathcal{F}|_V)$. But we also have $R^p(g \circ f)_*\mathcal{F}(U) = H^p(V, \mathcal{F}|_V)$. Since both sheaves are quasicoherent, this determines an isomorphism $R^p(g \circ f)_*\mathcal{F}|_U \cong R^p g_*(f_*\mathcal{F})|_U$. We obtain a global isomorphism by glueing over an affine open cover of Z.

Now suppose g is affine. Then $g_*(R^p f_*\mathcal{F})(U) = R^p f_*\mathcal{F}(g^{-1}(U))$. Since $g^{-1}(U)$ is also affine, we have

$$R^p f_*\mathcal{F}(g^{-1}(U)) = H^p(V, \mathcal{F}|_V) = R^p(g \circ f)_*\mathcal{F}(U).$$

This determines an isomorphism $R^p(g \circ f)_*\mathcal{F}|_U \cong g_*(R^p f_*\mathcal{F})|_U$, and we again finish by glueing.

3. Let X and Y be proper schemes over a noetherian ring A. Consider an invertible sheaf $\mathcal{L}$ on X. Prove:

   (a) If $\mathcal{L}$ is ample and $i\colon Y \hookrightarrow X$ is any closed embedding, then $i^*\mathcal{L}$ is ample.

   (b) The sheaf $\mathcal{L}$ is ample if and only if $i^*\mathcal{L}$ is ample for every reduced irreducible component $i\colon Y \hookrightarrow X$.

   (c) For any finite surjective morphism $f\colon Y \to X$, the sheaf $\mathcal{L}$ is ample if an only if $f^*\mathcal{L}$ is ample.

   Solution: (a) Since closed embeddings are finite, this is a corollary of Sheet 4, Exercise 4.

   (b) See [Liu, Corollary 5.3.8].

   (c) See [Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Proposition 1.4.4] or [Stacks, Tag 0B5V].

4. Let $X := \mathbb{P}^n_k$ for a field k.

(a) Show that for any integer $0 \leqslant q \leqslant n$ there is a short exact sequence

$$0 \to \Omega^q_{X/k} \to \mathcal{O}_X(-q)^{\binom{n+1}{q}} \to \Omega^{q-1}_{X/k} \to 0.$$

(b) Compute $\dim_k H^p(X, \Omega^q_{X/k})$ for $X := \mathbb{P}^n_k$ and all p, q.

Solution: (a) As preparation, we give the following lemma:

Lemma. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of locally free sheaves on a scheme X where $\mathcal{F}''$ has rank 1. Then there is an exact sequence*

$$0 \to \bigwedge^q \mathcal{F}' \to \bigwedge^q \mathcal{F} \to \Big(\bigwedge^{q-1} \mathcal{F}'\Big) \otimes \mathcal{F}'' \to 0.$$

The proof of the lemma is analogous to that of the earlier statement about the top exterior power from the course. Recall that we have an exact sequence

$$0 \to \Omega_{X/k} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0.$$

Applying the lemma to this yields the desired exact sequence.

(b) If $q \notin [0, n]$ then $\Omega^q_{X/k} = 0$ and hence $H^p(X, \Omega^q_{X/k}) = 0$. For $q = 0$ we have $\Omega^0_{X/k} = \mathcal{O}_X$ and know already that $H^0(X, \Omega^0_{X/k}) = k$ and $H^p(X, \Omega^0_{X/k}) = 0$ for all $p \neq 0$. For $1 \leqslant q \leqslant n$ the long exact cohomology sequence associated to the short exact sequence from (a) is

$$\cdots\cdots\cdots \longrightarrow H^{p-1}(X, \mathcal{O}_X(-q))^{\binom{n+1}{q}} \longrightarrow H^{p-1}(X, \Omega^{q-1}_{X/k}) \longrightarrow$$

$$\longrightarrow H^p(X, \Omega^q_{X/k}) \longrightarrow H^p(X, \mathcal{O}_X(-q))^{\binom{n+1}{q}} \longrightarrow \cdots\cdots\cdots$$

where we have also used that cohomology commutes with direct sums. Since $1 \leqslant q \leqslant n$, from the course we know that $H^p(X, \mathcal{O}_X(-q)) = 0$ for all $p \in \mathbb{Z}$. It follows from the above long exact sequence that $H^{p-1}(X, \Omega^{q-1}_{X/k}) \cong H^p(X, \Omega^q_{X/k})$. By induction on q, we thus obtain

$$\dim_k H^p(X, \Omega^q_{X/k}) \cong \begin{cases} 1 & \text{if } 0 \leqslant p = q \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

5. Let Y be the curve in $\mathbb{P}^3_k$ over an algebraically closed field k that is defined by the equations $X_0^2 + X_2^2 = aX_1X_3$ and $X_1^2 + X_3^2 = aX_0X_2$ for a constant $a \in k$. Compute $H^*(Y, \mathcal{O}_Y)$, find out when Y is regular, and in that case compute $H^*(Y, \Omega_{Y/k})$.

Solution: Since Y is one-dimensional and projective, all cohomology groups except perhaps H^0 and H^1 are zero. Let $X := \mathbb{P}^3_k = \operatorname{Proj} R$ for $R := k[X_0, \ldots, X_3]$.

(a) Regardless of the value of $a \in k$, the polynomials $f := X_0^2 + X_2^2 - aX_1X_3$ and $g := X_1^2 + X_3^2 - aX_0X_2$ are relatively prime. Thus we have an exact sequence of graded R-modules

$$0 \longrightarrow R(-4) \xrightarrow{\binom{g}{-f}} R(-2)^{\oplus 2} \xrightarrow{(f,g)} I := (f,g) \longrightarrow 0.$$

This induces an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X(-4) \xrightarrow{\binom{g}{-f}} \mathcal{O}_X(-2)^{\oplus 2} \xrightarrow{(f,g)} \mathcal{I} \longrightarrow 0.$$

Since $H^p(X, \mathcal{O}_X(-2)) = 0$ for all integers p, the associated long exact cohomology sequence yields an isomorphism $H^p(X, \mathcal{I}) \xrightarrow{\sim} H^{p+1}(X, \mathcal{O}_X(-4))$ for every p. The latter is $\cong k$ if $p + 1 = 3$ and 0 otherwise. We also have an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0$$

where $i\colon Y \hookrightarrow X$ is the given closed embedding. Since $H^p(X, \mathcal{O}_X) = 0$ for all $p > 0$, the associated long exact cohomology sequence yields isomorphisms

$$k = H^0(X, \mathcal{O}_X) \xrightarrow{\sim} H^0(X, i_*\mathcal{O}_Y) \cong H^0(Y, \mathcal{O}_Y)$$

and

$$H^1(Y, \mathcal{O}_Y) \cong H^1(X, i_*\mathcal{O}_Y) \xrightarrow{\sim} H^2(X, \mathcal{I}) \xrightarrow{\sim} H^3(X, \mathcal{O}_X(-4)) \cong k.$$

Thus $H^p(Y, \mathcal{O}_Y) \cong k$ for $p = 0, 1$ and $= 0$ otherwise.

(b) Since Y has codimension 2 and is given by two equations, we know that Y is regular if and only if the Jacobian has rank 2 everywhere on Y. By the symmetry of the defining equations, it is enough to determine when $Y \cap D_{X_0}$ is regular. Writing $x_i := X_i/X_0$ for $i = 1, 2, 3$, the Jacobian matrix of $(f/X_0^2, g/X_0^2)^T$ is

$$\begin{pmatrix} -ax_3 & 2x_2 & -ax_1 \\ 2x_1 & -a & 2x_3 \end{pmatrix}.$$

Thus the singular locus of $Y \cap D_{X_0}$ is the joint zero set in $\mathbb{A}_k^3$ of f/X_0^2 and g/X_0^2 and the three 2×2-minors of the Jacobian matrix, in other words

$$V\big(1 + x_2^2 - ax_1x_3,\ x_1^2 + x_3^2 - ax_2,\ a^2x_3 - 4x_1x_2,\ 4x_2x_3 - a^2x_1,\ 2a(x_1^2 - x_3^2)\big).$$

If $a = 0$, this contains the singular point $(x_1, x_2, x_3) = (0, \pm i, 0)$.

If $a \neq 0$ and $2 = 0$, the third and fourth polynomial imply that $x_3 = x_1 = 0$, which by the second polynomial implies that $x_2 = 0$, which yields a contradiction by the first polynomial.

If $2a \neq 0$, the last polynomial implies that $x_3 = \pm x_1$; then the second polynomial implies that $x_2 = 2x_1^2/a$. If $x_1 = 0$, this leads to a contradiction as in the previous case. Otherwise the third or fourth polynomial yields $x_2 = a^2x_3/4x_1 = \pm a^2/4$ and hence $x_1x_3 = \pm x_1^2 = \pm ax_2/2 = a^3/8$. Plugging everything into the first polynomial we obtain the equation $1 + \frac{a^4}{16} - \frac{a^4}{8} = 0$, which is equivalent to $a^4 = 16$. Conversely if $a^4 = 16$, one checks that $x_1 = x_3 = \sqrt{2/a}$ and $x_2 = a^2/4$ defines a singular point.

Together all this shows that the curve is non-singular if and only if $a \notin \{0\} \cup 2\mu_4$, where μ_4 denotes the set of fourth roots of unity.

(c) Again since f and g are relatively prime, we have an isomorphism of graded R-modules

$$(f,g)\colon (R/I)(-2)^{\oplus 2} \xrightarrow{\sim} I/I^2.$$

This induces an isomorphism $i^*\mathcal{O}_X(-2)^{\oplus 2} \cong i^*(\mathcal{I}/\mathcal{I}^2)$. The second exact sequence for differentials thus reads

$$0 \longrightarrow i^*\mathcal{O}_X(-2)^{\oplus 2} \longrightarrow i^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

These sheaves are locally free of respective ranks $2, 3, 1$. Taking highest exterior powers we obtain an isomorphism

$$i^*\mathcal{O}_X(-4) \;\cong\; \bigwedge^3 i^*\Omega_{X/k} \;\cong\; \Omega_{Y/k} \otimes \bigwedge^2 i^*\big(\mathcal{O}_X(-2)^{\oplus 2}\big) \;\cong\; \Omega_{Y/k} \otimes i^*\mathcal{O}_X(-4).$$

Canceling the factor $i^*\mathcal{O}_X(-4)$ shows that $\Omega_{Y/k} \cong \mathcal{O}_Y$. The formula for $H^p(Y, \Omega_{Y/k})$ is thus the same as that in (a).

(*Remark:* In a similar way we determined the canonical sheaf of any smooth hypersurface in $\mathbb{P}^n_k$ in §5.10 of the course. The method generalizes to any smooth *global complete intersection*, i.e., any smooth closed subvariety $Y \subset \mathbb{P}^n_k$ of codimension r that is given globally by r equations of degrees $d_1, \ldots, d_r$. The result is that $\omega_{Y/k} \cong i^*\mathcal{O}(d_1 + \ldots + d_r - n - 1)$.)

*6. (a) Let (C, d_C) and (D, d_D) be complexes of A-modules, where A is a ring. We define a complex $C \otimes D$ whose degree m part is given by

$$(C \otimes D)^m = \bigoplus_{p+q=m} C^p \otimes_A D^q$$

for every $m \in \mathbb{Z}$. The boundary maps d are given on each summand via $d(f \otimes g) := d_C f \otimes g + (-1)^{\deg f} f \otimes d_D g$, and we extend by linearity. We call $\big(C \otimes D, d\big)$ the *tensor product* of C and D. Show that if A is a field, we have a natural isomorphism

$$H^m\big(C \otimes D\big) \cong \bigoplus_{p+q=m} H^p(C) \otimes_A H^q(D).$$

(b) (*Künneth Formula.*) Let X and Y be quasi-compact separated schemes over a field k, and let $\mathcal{F}$ and $\mathcal{G}$ be quasicoherent sheaves on X and Y respectively. Show that for every $m \geqslant 0$, there is a natural isomorphism

$$H^m(X \times_k Y, \mathrm{pr}_X^*\mathcal{F} \otimes \mathrm{pr}_Y^*\mathcal{G}) \cong \bigoplus_{p+q=m} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}).$$

Solution: (a) Suppose A is a field. We regard the graded modules $Z(C)$ and $B(C)$ as chain complexes with zero differential. The exact sequences

$$0 \longrightarrow Z^p(C) \longrightarrow C^p \xrightarrow{d_C} B^{p+1}(C) \longrightarrow 0$$

yield a long exact sequence of complexes

$$0 \longrightarrow Z(C) \longrightarrow C \xrightarrow{d_C} B(C) \longrightarrow 0.$$

Since we are over a field, tensoring (of complexes) preserves exactness, and we obtain an exact sequence

$$0 \longrightarrow Z(C) \otimes D \longrightarrow C \otimes D \longrightarrow B(C) \otimes D \longrightarrow 0. \tag{2}$$

Using that the boundary maps in $Z(C)$ and $B(C)$ are zero, a simple computation shows that for every $m \in \mathbb{Z}$, we have

$$H^m\big(Z(C) \otimes D\big) = \bigoplus_{p+q=m} Z^p(C) \otimes H^q(D) =: \big(Z(C) \otimes H^*(D)\big)^m$$

and

$$H^m\big(B(C) \otimes D\big) = \bigoplus_{p+q=m} B^{p+1}(C) \otimes H^q(D) =: \big(B(C) \otimes H^*(D)\big)^m.$$

Furthermore, one sees by inspection of the definitions that the connecting homomorphism of the long exact sequence of cohomology associated to (2) is just the inclusion $\big(B(C) \otimes H^*(D)\big)^* \hookrightarrow \big(Z(D) \otimes H^*(D)\big)^*$. This means that the long exact sequence breaks up into short exact sequences

$$0 \longrightarrow \big(B(C) \otimes H^*(D)\big)^{m-1} \longrightarrow \big(Z(C) \otimes H^*(D)\big)^m \longrightarrow H^m(C \otimes D) \longrightarrow 0.$$

Since tensoring with $H^*(D)$ is exact, the cokernel of $\big(B(C) \otimes H^*(D)\big)^{m-1} \hookrightarrow \big(Z(D) \otimes H^*(D)\big)^m$ is precisely $\bigoplus_{p+q=m} H^p(C) \otimes_A H^q(D)$, and the desired result follows.

(b) Let $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ be affine open coverings of X and Y respectively. Then $\mathcal{U} \times \mathcal{V} := (U_i \times_k V_j)_{(i,j) \in I \times J}$ is an affine open covering of $X \times_k Y$. For the Čech complexes we have

$$C^*(\mathcal{U} \times \mathcal{V}, \mathrm{pr}_X^*\mathcal{F} \otimes \mathrm{pr}_Y^*\mathcal{G}) = C^*(\mathcal{U}, \mathcal{F}) \otimes C^*(\mathcal{V}, \mathcal{G}).$$

The desired result then follows directly from part (a) and the fact that Čech cohomology can be computed with affine open coverings in this situation.