Solutions 11

Higher Direct Image, Duality, Base Change

1. Let $f\colon X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{L}$ be a relatively ample invertible sheaf on X over Y, and let $\mathcal{F}$ be a coherent sheaf on X. Show:

   (a) For all $n \gg 0$, the natural map $f^*f_*\big(\mathcal{F}\otimes\mathcal{L}^{\otimes n}\big) \to \mathcal{F}\otimes\mathcal{L}^{\otimes n}$ is surjective.

   (b) For $p > 0$ and $n \gg 0$, we have $R^pf_*(\mathcal{F}\otimes\mathcal{L}^{\otimes n}) = 0$.

   Solution: (a) Let $U = \operatorname{Spec} A \subset Y$ be open and let $V := f^{-1}(U)$. We know from the course that
   $$f_*(\mathcal{F}\otimes\mathcal{L}^{\otimes n})|_U \cong H^0(V, (\mathcal{F}\otimes\mathcal{L}^{\otimes n})|_V)^{\sim}.$$
   Then the homomorphism being surjective is equivalent to saying that $(\mathcal{F}\otimes\mathcal{L}^{\otimes n})|_V$ is generated by its global sections, but this is true for large n since $\mathcal{L}|_V$ is ample. Since Y is quasicompact, we may choose a finite affine open covering $Y = \bigcup_{i=1}^n U_i$ and n large enough so that the restrictions to each U_i are surjective. This yields the desired result.

   (b) If Y is affine, this translates into $H^p(X, \mathcal{F}\otimes\mathcal{L}^{\otimes n}) = 0$ for all $p > 0$ and $n \gg 0$, which is the cohomological criterion for ampleness. Choosing a finite open affine covering of Y and n large enough to work for each member of the covering as in part (a) yields the desired result.

2. Show the following:

   (a) For any flat morphism $f\colon X \to Y$ the functor f^* from the category of $\mathcal{O}_Y$-modules to the category of $\mathcal{O}_X$-modules is exact.

   (b) For any morphism $f\colon X \to Y$ and any flat $\mathcal{O}_Y$-module $\mathcal{G}$ the $\mathcal{O}_X$-module $f^*\mathcal{G}$ is flat.

   (c) For any flat morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ the composite $g \circ f$ is flat.

   (d) For any flat morphism $f\colon X \to Y$ and any morphism $g\colon Y' \to Y$ the morphism $X \times_Y Y' \to Y'$ is flat.

   Solution: (a) Let
   $$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{1}$$
   be an exact sequence of $\mathcal{O}_Y$-modules and consider the sequence
   $$0 \to f^*\mathcal{F}' \to f^*\mathcal{F} \to f^*\mathcal{F}'' \to 0 \tag{2}$$

of $\mathcal{O}_X$-modules. Let $x \in X$, and let $y := f(x)$. On stalks in (2) we have

$$0 \to \mathcal{F}'_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{F}''_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to 0.$$

Since (1) is exact, hence exact on stalks, and $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module, this is an exact sequence of $\mathcal{O}_{X,x}$-modules. Since exactness can be checked on stalks, this implies that (2) is exact.

(b) Let x and y be as above. Then $(f^*\mathcal{G})_x \cong \mathcal{G}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$. We need to show that $(f^*\mathcal{G})_x$ is a flat $\mathcal{O}_{X,x}$-module, which results from the following lemma:

Lemma 1. *Let $A \to B$ be a homomorphism of rings and let M be a flat A-module. Then $M \otimes_A B$ is a flat B-module.*

Proof. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of B-modules. Tensoring by $B \otimes_A M$ and contracting the tensor product, we obtain

$$0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0,$$

which is exact since M is flat. The desired result follows. □

(c) Let x, y be as before and let $z := g(y)$. We know that $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$, which is flat over $\mathcal{O}_{Z,z}$. We just need to show that $\mathcal{O}_{X,x}$ is then flat over $\mathcal{O}_{Z,z}$. This follows from

Lemma 2. *Let $A \to B \to C$ be ring homomorphisms such B is flat over A and C is flat over B. Then C is flat over A.*

Proof. Since the functor $\otimes_A C$ is isomorphic to the composition of the exact functors $\otimes_A B$ and $\otimes_B C$, it is exact as well. Thus C is flat over A. □

(d) Let $X' := X \times_Y Y'$. Let $x' \in X'$ with images $y' \in Y'$ and $x \in X$ and $y \in Y$. Then $\mathcal{O}_{X',x'} \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y',y'}$ and is thus flat over $\mathcal{O}_{Y,y'}$ by Lemma 2.

**3. Show that every smooth morphism is flat.

Solution: See [Görtz and Wedhorn, Theorem 14.22].

4. Let Y be locally noetherian and consider a projective morphism $f: X \to Y$ with r-dualizing sheaf ω_f. Show that for any flat morphism $Y' \to Y$ from a locally noetherian scheme Y', the dualizing sheaf of $X \times_Y Y' \to Y'$ is isomorphic to $\mathrm{pr}_X^* \omega_f$.

Solution: We separate the proof into steps:

(I) Let $g: Y' \to Y$ denote the morphism from the exercise. Applying g^* to the trace map $\mathrm{tr}_f: R^r f_* \omega_f \to \mathcal{O}_Y$, we obtain a morphism

$$g^*(\mathrm{tr}_f): g^* R^r f_* \omega_f \to \mathcal{O}_{Y'}.$$

Let $f' := \mathrm{pr}_Y$ and $g' := \mathrm{pr}_X$. Since g is flat, the base change homomorphism

$$g^* R^r f_* \omega_f \to R^r f'_* g'^* \omega_f$$

is an isomorphism. Precomposing $g^*(\mathrm{tr}_f)$ with the base change isomorphism, we obtain a homomorphism

$$tr_{f'} \colon R^r f'_* g'^* \omega_f \to \mathcal{O}_{Y'},$$

which we claim makes $g'^* \omega_f$ into an r-dualizing sheaf for f'. We need the following lemma:

Lemma 3. *Let $f\colon X \to Y$ be a flat morphism of schemes and let $\mathcal{G}$ and $\mathcal{G}'$ be $\mathcal{O}_Y$-modules, with $\mathcal{G}$ of finite presentation. Then the natural homomorphism $\alpha\colon f^* \mathscr{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{G}') \to \mathscr{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, f^*\mathcal{G}')$ is an isomorphism.*

Proof. Let $x \in X$ and let $y := f(x)$. Taking stalks and using [Görtz and Wedhorn, Propositon 7.16], the homomorphism α yields the natural homomorphism

$$\mathrm{Hom}_{\mathcal{O}_{Y,y}}(\mathcal{G}_y, \mathcal{G}'_y) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}, \mathcal{G}'_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}).$$

It remains to show that this is an isomorphism, which is a consequence of the following fact from Commutative Algebra: Let R be a ring and let M and N be R-modules. Let R' be a flat R-algebra. Then the natural homomorphism

$$\mathrm{Hom}_R(M, N) \otimes R' \to \mathrm{Hom}_{R'}(M \otimes_R R', N \otimes_R R')$$

is an isomorphism. For a proof of this, see [Stacks, Tag 087R]. □

(II) Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. By assumption, $(\omega_f, \mathrm{tr}_f)$ yields an isomorphism

$$f_* \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_Y}(R^r f_* \mathcal{F}, \mathcal{O}_Y).$$

Applying g^* and the lemma we obtain an isomorphism

$$g^* f_* \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}'_Y}(g^* R^r f_* \mathcal{F}, \mathcal{O}_{Y'}).$$

Since g is flat, the base change homomorphism is an isomorphism. Since g' is also flat by Exercise 2d, we can combine this with the lemma to obtain

$$f'_* \mathscr{H}om_{\mathcal{O}_{X'}}(g'^* \mathcal{F}, g'^* \omega_f) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}'_Y}(R^r f'_* g'^* \mathcal{F}, \mathcal{O}_{Y'}).$$

By construction, this is precisely the morphism induced by $(g'^* \omega_f, \mathrm{tr}_{f'})$. The pair $(g'^* \omega_f, \mathrm{tr}_{f'})$ thus satisfies the r-dualizing sheaf condition for sheaves of the form $g'^* \mathcal{F}$, where $\mathcal{F}$ is coherent.

(III) Let $\mathcal{F}'$ be an arbitrary coherent sheaf on X'. Since $\mathcal{O}_{X'} \cong g'^*\mathcal{O}_X$, we have a presentation of the form $g'^*\mathcal{O}_X^n \to g'^*\mathcal{O}_X^m \to \mathcal{F}' \to 0$. By functoriality, we obtain a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
f'_*\mathscr{H}om_{\mathcal{O}_{X'}}(\mathcal{F}', g'^*\omega_f) & \longrightarrow & \mathscr{H}om_{\mathcal{O}'_Y}(R^r f'_*\mathcal{F}', \mathcal{O}_{Y'}) \\
\downarrow & & \downarrow \\
f'_*\mathscr{H}om_{\mathcal{O}_{X'}}(g'^*\mathcal{O}_X^m, g'^*\omega_f) & \xrightarrow{\sim} & \mathscr{H}om_{\mathcal{O}'_Y}(R^r f'_* g'^*\mathcal{O}_X^m, \mathcal{O}_{Y'}) \\
\downarrow & & \downarrow \\
f'_*\mathscr{H}om_{\mathcal{O}_{X'}}(g'^*\mathcal{O}_X^n, g'^*\omega_f) & \xrightarrow{\sim} & \mathscr{H}om_{\mathcal{O}'_Y}(R^r f'_*\mathcal{O}_X^n, \mathcal{O}_{Y'})
\end{array}$$

Since the Sheaf-Hom and pushforward are both left exact, the left column is exact.

Recall that the r-dualizing sheaf for $f\colon X \to Y$ is defined for f projective and Y locally noetherian and such that all fibers of f have dimension $\leqslant r$. Since f' is the base change of f, it satisfies the same properties. By [Liu, Propositon 5.2.34], this implies that $R^p f'_*\mathcal{G}' = 0$ for every $p > r$ and every quasicoherent sheaf $\mathcal{G}'$ on X'. Hence $R^r f'_*$ is right exact. Thus both columns in the above diagram are exact, and we conclude that the top arrow is an isomorphism by the Five Lemma. The pair $(g'^*\omega_f, \mathrm{tr}_{f'})$ thus satisfies the r-dualizing sheaf condition for coherent sheaves.

(IV) For a general quasi-coherent sheaf $\mathcal{F}'$ on X', we write $\mathcal{F}' = \cup_{i\in I}\mathcal{F}'_i$ for a filtered direct system of coherent sheaves $\mathcal{F}_i$, and proceed exactly as in the proof of the theorem regarding the dualizing sheaf of $\mathbb{P}^n_Y$ from §6.5 of the course.

5. (*Projection Formula*) (Compare Sheet 3, Exercise 2) Consider a morphism $f\colon X \to Y$, an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$.

   (a) Construct a natural base change homomorphism $(R^p f_*\mathcal{F}) \otimes \mathcal{G} \to R^p f_*(\mathcal{F} \otimes f^*\mathcal{G})$.

   (b) If f is separated and quasi-compact and $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent and $\mathcal{G}$ is flat, then this is an isomorphism.

   Solution: See [Liu, Proposition 5.2.32].

6. Let $Y = \operatorname{Spec} A$ and $X = \operatorname{Proj} A[X,Y] = \mathbb{P}^1_A$. Let $\mathcal{F}$ be the kernel of the homomorphism $\varphi := (X^2, aXY, Y^2)\colon \mathcal{O}_X^{\oplus 3} \to \mathcal{O}_X(2)$ for some $a \in A$.

   (a) Show that $\mathcal{F}$ is locally free and the sequence $0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus 3} \xrightarrow{\varphi} \mathcal{O}_X(2) \to 0$ is exact.

   (b) For every integer p compute $H^p(X, \mathcal{F})$.

   (c) For every point $y \in Y$ with fiber $i_y\colon X_y \hookrightarrow X$ and every integer p compute $H^p(X_y, i_y^*\mathcal{F})$ and compare it with $H^p(X, \mathcal{F}) \otimes_A k(y)$.

Solution: (a) We know that X^2 and Y^2 generate $\mathcal{O}_X(2)$, from which it follows that φ is surjective and hence that the sequence is exact. Since $\mathcal{O}_X(2)$ is locally free, the sequence is locally split. This implies that the stalks of $\mathcal{F}$ are free and hence that $\mathcal{F}$ is locally free.

(b) In what follows, we abbreviate $H^p(X, \mathcal{F})$ by $H^p(\mathcal{F})$ for all $p \in \mathbb{Z}$. Since X is a curve, we already know that $H^p(\mathcal{F}) = 0$ for $p \neq 0, 1$. The long exact sequence associated to the short exact sequence in (a) yields

$$\begin{array}{ccccccccc} 0 \longrightarrow H^0(\mathcal{F}) \longrightarrow & H^0(\mathcal{O}_X^{\oplus 3}) & \longrightarrow & H^0(\mathcal{O}_X(2)) & \longrightarrow & H^1(\mathcal{F}) \longrightarrow & H^1(\mathcal{O}_X^{\oplus 3}) \\ & \| & & \| & & & \| \\ & A^{\oplus 3} & \xrightarrow{\varphi} & A \cdot X^2 \oplus A \cdot XY \oplus A \cdot Y^2 & & & 0. \end{array}$$

After splitting off $A \oplus \{0\} \oplus A$ in the second term and $AX^2 \oplus AY^2$ in the third we obtain an exact sequence

$$0 \longrightarrow H^0(\mathcal{F}) \longrightarrow A \xrightarrow{a\cdot} A \longrightarrow H^1(\mathcal{F}) \longrightarrow 0.$$

We thus have

$$\begin{aligned} H^0(\mathcal{F}) &\cong \ker(a \colon A \to A), \\ H^1(\mathcal{F}) &\cong \operatorname{coker}(a \colon A \to A) = A/aA. \end{aligned}$$

(c) Let $y \in Y$. Let $X_y := X \times_Y \operatorname{Spec} k(y)$ with the canonical closed embedding $i_y \colon X_y \hookrightarrow X$. Then the calculation (b) with $k(y), a(y)$ in place of A, a shows that

$$H^p(X_y, i_y^*\mathcal{F}) \cong \begin{cases} 0 & \text{if } a(y) \neq 0 \text{ or } p \neq 0, 1, \\ k(y) & \text{if } a(y) = 0 \text{ and } p = 0, 1. \end{cases}$$

For $p = 1 = \dim(X)$ we know from the lecture that the base change homomorphism $H^p(X, \mathcal{F}) \otimes_A k(y) \to H^p(X_y, i_y^*\mathcal{F})$ is an isomorphism, and we see it again concretely because $(A/aA) \otimes_A k(y) \cong k(y)/(a(y))$. But for $p = 0$ it could be that A is an integral domain and a is a non-zero element of the prime ideal associated to y, in which case $H^0(X, \mathcal{F}) = 0$ but $H^0(X_y, i_y^*\mathcal{F}) \neq 0$, so the base change homomorphism cannot be an isomorphism.