Algebraic Geometry II

HS 2017

Solutions 12

Euler Characteristic, Riemann-Roch, Residues

1. (*Riemann-Roch for locally free sheaves*) Let X be a connected smooth projective curve of genus g over an algebraically closed field k.

   (a) For every non-zero locally free sheaf $\mathcal{F}$ there exists an invertible sheaf $\mathcal{L} \subset \mathcal{F}$ such that $\mathcal{F}/\mathcal{L}$ is locally free.

   (b) For any locally free sheaf $\mathcal{F}$ of rank r over X define $\deg(\mathcal{F}) := \deg(\bigwedge^r \mathcal{F})$ and prove that
$$\chi(X, \mathcal{F}) \;=\; r \cdot (1-g) + \deg(\mathcal{F}).$$

Solution: (a) Let $U \subset X$ be a nonempty open subscheme such that $\mathcal{F}|_U$ is free and let $j\colon U \hookrightarrow X$ be the canonical embedding. Since X is integral, the adjunction map $\mathcal{F} \to j_*j^*\mathcal{F}$ is injective; we identify $\mathcal{F}$ with its image. Choose a direct summand $\mathcal{M} \subset j^*\mathcal{F}$ which is free of rank 1. Since j_* is left exact, we have $j_*\mathcal{M} \subset j_*j^*\mathcal{F}$. We claim that $\mathcal{L} := \mathcal{F} \cap j_*\mathcal{M}$ has the desired property.

For this recall that the local rings of X are principal ideal domains, and that a finitely generated module over a principal ideal domain is free if and only if it is torsion free. Since $\mathcal{L}$ is a subsheaf of $\mathcal{F}$, it is finitely generated and torsion free, and we thus deduce that $\mathcal{L}$ is locally free. Moreover by construction $\mathcal{L}|_U \cong \mathcal{M}$, and so $\mathcal{L}$ has rank 1. Also by construction and the exactness of j_* we have
$$\mathcal{F}/\mathcal{L} \;=\; \mathcal{F}/(\mathcal{F} \cap j_*\mathcal{M}) \;\hookrightarrow\; j_*j^*\mathcal{F}/j_*\mathcal{M} \;\cong\; j_*(j^*\mathcal{F}/\mathcal{M}).$$

Being the j_* of a free sheaf the right hand side is torsion free; hence $\mathcal{F}/\mathcal{L}$ is torsion free. But $\mathcal{F}/\mathcal{L}$ is also a quotient of $\mathcal{F}$; hence finitely generated; hence locally free.

(b) We proceed by induction on r. If $r = 0$, the sheaf $\bigwedge^r \mathcal{F} = \mathcal{O}_X$ has degree 0 and both sides are 0. If $r = 1$, the statement is version 1 of the Riemann-Roch theorem from the course. Suppose $r > 1$. Let $\mathcal{L}$ be as in part (a) and set $\mathcal{E} := \mathcal{F}/\mathcal{L}$. Then we have an exact sequence $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{E} \to 0$. Since rank is additive, we see that $\mathcal{E}$ has rank $r-1$. By the induction hypothesis, we have
$$\begin{aligned}\chi(X,\mathcal{F}) = \chi(X,\mathcal{L}) + \chi(X,\mathcal{E}) &= (1-g) + \deg(\mathcal{L}) + (r-1)(1-g) + \deg(\mathcal{E}) \\ &= r(1-g) + \deg(\mathcal{L}) + \deg(\mathcal{E}).\end{aligned}$$

Since the exact sequence above induces an isomorphism $\bigwedge^r \mathcal{F} \cong \bigwedge^{r-1} \mathcal{E} \otimes \mathcal{L}$, we deduce that $\deg(\mathcal{L}) + \deg(\mathcal{E}) = \deg(\mathcal{F})$, which yields the desired result.

2. For an arbitrary integral projective curve X over an algebraically closed field k, the *arithmetic genus of* X is defined as $p_a(Y) := h^1(X, \mathcal{O}_X)$. Let $\pi\colon \tilde{X} \to X$ be the normalization of X.

   (a) Show that $p_a(X) = p_a(\tilde{X}) + \sum'_{P\in X} \text{length}_{\mathcal{O}_{X,P}}(\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)_P$.

   (b) Deduce that $p_a(X) = 0$ if and only if X is nonsingular of genus 0.

   (c) Determine $p_a(X)$ for the nodal cubic curve $X := V(C(C-B)A - B^3) \subset \mathbb{P}^2_k$ and the cuspidal cubic curve $X := V(B^2C - A^3) \subset \mathbb{P}^2_k$.

   Solution: (a) The morphism π is birational, and it is finite because X is noetherian. Thus $\pi_*\mathcal{O}_{\tilde{X}}$ is a coherent sheaf of $\mathcal{O}_X$-modules and the homomorphism $\pi^\flat\colon \mathcal{O}_X \to \pi_*\mathcal{O}_{\tilde{X}}$ is an isomorphism except at the finitely many closed points where X is singular. Since X is integral, it follows that $\pi^\flat$ is injective everywhere. We will study the long exact cohomology sequence associated to the short exact sequence $0 \to \mathcal{O}_X \to \pi_*\mathcal{O}_{\tilde{X}} \to (\pi_*\mathcal{O}_{\tilde{X}})/\mathcal{O}_X \to 0$.

   Since π is affine, we have natural isomorphism $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^i(X, \pi_*\mathcal{O}_{\tilde{X}})$ for each i. As X and $\tilde{X}$ are integral projective, we have $H^0(X, \mathcal{O}_X) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = k$. Since $\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ is a coherent sheaf with finite support, the group of its global sections is the direct sum of its stalks and its H^1 vanishes. The long exact cohomology sequence thus reads

$$0 \longrightarrow k \longrightarrow k \xrightarrow{(*)} \bigoplus_{P\in X} (\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)_P \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow 0.$$

   The homomorphism $(*)$ must therefore vanish, and taking dimensions yields the desired formula in (a).

   (b) Every term on the right hand side of the equation in (a) is non-negative. Thus $p_a(X) = 0$ if and only if $p_a(\tilde{X}) = 0$ and $\mathcal{O}_X = \pi_*\mathcal{O}_{\tilde{X}}$. The latter is equivalent to $X = \tilde{X}$ (for instance because it means that each stalk $\mathcal{O}_{X,P}$ is integrally closed). Thus $p_a(X) = 0$ if and only if X is non-singular of genus 0.

   (c) Each curve has a closed embedding $i\colon X \hookrightarrow \mathbb{P}^2_k$ as a curve of degree 3; so we have a short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2_k}(-3) \to \mathcal{O}_{\mathbb{P}^2_k} \to i_*\mathcal{O}_X \to 0$, regardless of singularities. The associated long exact cohomology sequence

$$\begin{array}{ccccccc} H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) & \longrightarrow & H^1(\mathbb{P}^2_k, i_*\mathcal{O}_X) & \longrightarrow & H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-3)) & \longrightarrow & H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \\ \| & & \wr\| & & \wr\| & & \| \\ 0 & & H^1(X, \mathcal{O}_X) & & k & & 0 \end{array}$$

   shows that $p_a(X) = h^1(X, \mathcal{O}_X) = 1$.

   Aliter: By earlier computations each curve has normalization $\tilde{X} \cong \mathbb{P}^1_k$ and precisely one singular point P. Show that $\dim_k(\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)_P = 1$ by an explicit local calculation and use (a). The local calculation amounts to the fact that

$$\dim_k(k[t]/k[t^2, t^2(1+t)]) = \dim_k(k[t]/k[t^2, t^3]) = 1.$$

3. (*Hilbert polynomial of a coherent sheaf*) Let X be a projective scheme over a field k with a very ample invertible sheaf $\mathcal{L}$ and an arbitrary coherent sheaf $\mathcal{F}$. Prove:

   (a) There is a unique polynomial $P_{\mathcal{F}} \in \mathbb{Q}[T]$ such that $\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = P_{\mathcal{F}}(m)$ for all $m \in \mathbb{Z}$.

   (b) This polynomial can be written uniquely as $P_{\mathcal{F}}(T) = \sum_n' a_n \binom{T}{n}$ with $a_n \in \mathbb{Z}$.

   *(c) If $\mathcal{F} \neq 0$, the degree of $P_{\mathcal{F}}$ is equal to the dimension of the support of $\mathcal{F}$ and the highest coefficient of $P_{\mathcal{F}}$ is positive.

   (d) If X is a smooth connected curve and k is algebraically closed, the highest coefficient of $P_{\mathcal{O}_X}$ is $\deg(\mathcal{L})$.

   *(e) Repeat the same with an arbitrary invertible sheaf $\mathcal{L}$, assuming only in (c) that $\mathcal{L}$ is ample.

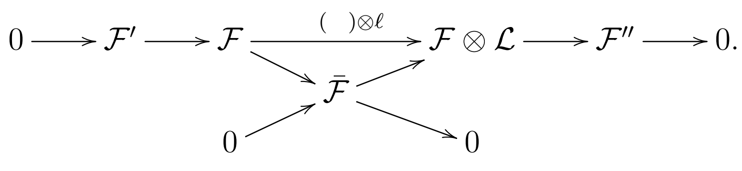
Solution: First note that for any field extension L/k and the base change morphism $\pi \colon X_L \to X$, we have $\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = \chi(X_L, \pi^*\mathcal{F} \otimes (\pi^*\mathcal{L})^{\otimes m})$. In all parts of the exercise we may thus assume that k is algebraically closed.

The uniqueness in (a) and (b) is a direct consequence of the fact that a univariate polynomial is determined by its values at any infinite set of points.

Recall from §5.6 that there is a unique smallest closed subscheme Y with embedding $i : Y \hookrightarrow X$ such that $\mathcal{F} \xrightarrow{\sim} i_*i^*\mathcal{F}$, called the scheme-theoretic support of $\mathcal{F}$. For every $m \in \mathbb{Z}$ we then have $\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = \chi(Y, i^*\mathcal{F} \otimes (i^*\mathcal{L})^{\otimes m})$. Thus we may reduce ourselves to the case that $Y = X$.

We then do induction on $d := \dim X$. If X is finite, we have $\mathcal{L} \cong \mathcal{O}_X$ and hence $H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \cong \Gamma(X, \mathcal{F})$ and all other cohomology groups are zero. Thus $\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$ is an integer that is independent of m and > 0 if $\mathcal{F} \neq 0$; proving (a) through (c) in this case.

If $d > 0$, let $i \colon X \hookrightarrow \mathbb{P}^N_k$ be a closed embedding with $i^*\mathcal{O}(1) \cong \mathcal{L}$. Since k is algebraically closed, there exists a hyperplane $L \subset \mathbb{P}^N_k$ containing none of the irreducible components of $i(X)$. The linear form defining L yields a section $\ell \in \mathcal{L}(X)$ which generates $\mathcal{L}$ over an open dense subset $U \subset X$. Define coherent sheaves $\mathcal{F}'$, $\mathcal{F}''$, and $\bar{\mathcal{F}}$ on X by the exact sequences



After tensoring with $\mathcal{L}^{\otimes m}$ the additivity of the Euler characteristic in short exact sequences shows that

$$\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m+1}) - \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \;=\; \chi(X, \mathcal{F}'' \otimes \mathcal{L}^{\otimes m}) - \chi(X, \mathcal{F}' \otimes \mathcal{L}^{\otimes m}).$$

Also $\mathcal{F}'$ and $\mathcal{F}''$ vanish on U, so their scheme-theoretic support has dimension $\leqslant d-1$; so by the induction hypothesis the right hand side is a polynomial of degree $\leqslant d-1$ in m. Write it in the form $\sum_{n=0}^{d-1} a_n\binom{m}{n}$. Since $\binom{m}{n} = \binom{m+1}{n+1} - \binom{m}{n+1}$, it follows that

$$\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m+1}) - \sum_{n=0}^{d-1} a_n \binom{m+1}{n+1} \;=\; \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) - \sum_{n=0}^{d-1} a_n \binom{m}{n+1}.$$

This value is therefore independent of m, and calling it b we deduce that

$$(*) \qquad \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \;=\; \sum_{n=1}^{d} a_{n-1}\binom{m}{n} + b$$

for all m. This proves both (a) and (b).

To do the induction step for (c) we claim that we can choose ℓ such that $\mathcal{F}' = 0$. This involves associated primes for modules, or in more down to earth terms: Cover X by finitely many open affines $\operatorname{Spec} A_i$ and suppose that the $\mathcal{F}|_{\operatorname{Spec} A_i} = \tilde{M}_i$. Then M_i is a finitely generated A_i-module; hence there exists a sequence of submodules $0 = M_{i,0} \subset M_{i,1} \subset \ldots \subset M_{i,r_i}$ such that each $M_{i,j}/M_{i,j-1} \cong A_i/\mathfrak{a}_{i,j}$ for some ideal $\mathfrak{a}_{i,j}$. Each associated prime of $\mathfrak{a}_{i,j}$ corresponds to an irreducible subset of $\operatorname{Spec} A_i$, whose closure is an irreducible subset of X. Let $\mathcal{S}$ denote the set of all irreducible closed subsets of X obtained in this way for all i and j. If the hyperplane L chosen above does not contain any of the irreducible subsets in $\mathcal{S}$, multiplication by ℓ is injective on each $A_i/\mathfrak{a}_{i,j}$ and hence on each M_i; from which one can deduce that $\mathcal{F}' = 0$. (Compare the discussion following [Vakil, Exercise 18.6.A].)

By contrast, recall that the support of $\mathcal{F}$ is X. Thus the support of $\mathcal{F}''$ is $X \cap L$. Also we have $d = \dim X > 0$. Moreover L meets every irreducible component of X of dimension d and the intersection has dimension $d-1$ by Krull's principal ideal theorem. Thus the support of $\mathcal{F}''$ has dimension $d-1$. By the induction hypothesis we thus have $\chi(X, \mathcal{F}'' \otimes \mathcal{L}^{\otimes m}) = \sum_{n=0}^{d-1} a_n\binom{m}{n}$ with $a_{d-1} > 0$. The formula $(*)$ above then shows that $\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = \sum_{n=1}^{d} a_{n-1}\binom{m}{n} + b$ with highest coefficient > 0, as desired.

In the situation of (d) observe that, with the definition of $\deg(\mathcal{L})$ from the course, by Riemann-Roch we have

$$\chi(X, \mathcal{L}^{\otimes m}) \;=\; 1 - g + \deg(\mathcal{L}^{\otimes m}) \;=\; 1 - g + \deg(\mathcal{L}) \cdot m.$$

Thus $P_{\mathcal{O}_X}(T) = 1 - g + \deg(\mathcal{L}) \cdot T$, which implies (e).

For (e) consider an arbitrary invertible sheaf $\mathcal{L}$. Choose an auxiliary very ample invertible sheaf $\mathcal{L}_1$ such that $\mathcal{L}_2 := \mathcal{L} \otimes \mathcal{L}_1$ is also very ample. Show in a similar fashion that $\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2})$ is a polynomial of total degree $\leqslant \dim(X)$ in (m_1, m_2). The special case $(m_1, m_2) = (-m, m)$ then yields everything except the positivity (c). For that assume that $\mathcal{L}$ is ample and choose $n > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample. Then the polynomial for $\mathcal{L}^{\otimes n}$ in place of $\mathcal{L}$ is obtained from that for $\mathcal{L}$ by substituting nT for T. This leaves the sign of the highest coefficient unchanged.

4. Let k be a field. Show that for any $f \in k((t))^\times$ and any $n \in \mathbb{Z}$ we have

$$\operatorname{res}_t(f^n df) \;=\; \begin{cases} \operatorname{ord}_t(f) & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: If $\operatorname{ord}_t(f) = 0$, then $\operatorname{ord}_t(f^n \frac{df}{dt}) \geqslant 0$ for all n; hence $\operatorname{res}_t(f^n df) = 0$. If $\operatorname{ord}_t(f) > 0$, the differential $f^n df = f^n \frac{df}{dt} dt$ arises by the substitution $s = f(t)$ from the differential $s^n ds$. By Proposition 2 of §7.3 of the course it follows that

$$\operatorname{res}_t(f^n df) = \operatorname{res}_s\big(\operatorname{tr}_{k((t))/k((s))}(s^n ds)\big) = \operatorname{res}_s\big(\operatorname{ord}_t(f) \cdot s^n ds\big) = \operatorname{ord}_t(f) \cdot \delta_{n,-1}.$$

If $\operatorname{ord}_t(f) < 0$, write $f = g^{-1}$; then the differential $f^n df = g^{-n} dg^{-1} = -g^{-n-2} dg$ arises by the substitution $s = g(t)$ from the differential $-s^{-n-2} ds$. By Proposition 2 of §7.3 of the course it follows that

$$\begin{aligned} \operatorname{res}_t(f^n df) &= \operatorname{res}_s\big(\operatorname{tr}_{k((t))/k((s))}(-s^{-n-2} ds)\big) \;=\; \operatorname{res}_s\big(\operatorname{ord}_t(g) \cdot (-s^{-n-2} ds)\big) \\ &= -\operatorname{ord}_t(g) \cdot \delta_{-n-2,-1} \;=\; \operatorname{ord}_t(f) \cdot \delta_{n,-1}. \end{aligned}$$

5. Let k be an algebraically closed field of characteristic $\neq 2$. Let X be the connected smooth projective curve over k with the affine equation $y^2 = f(x)$ for a separable polynomial $f(x) \in k[x]$ of degree 3. Denote the function field of X by K.
   (a) Show that $\Gamma(X, \Omega_{X/k}) = k \cdot \frac{dx}{y}$.
   (b) Verify the residue theorem for the rational differentials dx, $\frac{dx}{x}$, $\frac{x\,dx}{y} \in \Omega_{K/k}$ by explicitly computing all residues.

Solution: Write $f(x) = (x - e_1)(x - e_2)(x - e_3)$ for $e_i \in k$ distinct. By the jacobian criterion the affine curve $U := \operatorname{Spec} k[X,Y]/(Y^2 - f(X))$ is non-singular. Thus U is an affine open chart in X, where x, y are the residue classes of X, Y. The closure of U under the standard embedding $U \hookrightarrow \mathbb{A}^2_k \hookrightarrow \mathbb{P}^2_k$ is given by the homogeneous equation $Y^2 Z = (X - e_1 Z)(X - e_2 Z)(X - e_3 Z)$, which again by the jacobian criterion is non-singular and hence isomorphic to X. From this equation we see that $X \smallsetminus U$ consists of the single point in projective coordinates $(0:1:0)$, which we denote simply as $\widetilde{\infty} \in X$. A local equation for X near $\widetilde{\infty}$ is obtained by substituting $x = s^{-1}$ and $y = ts^{-2}$, resulting in the equation $t^2 = g(s)$ with $g(s) := s(1 - e_1 s)(1 - e_2 s)(1 - e_3 s)$, where $\widetilde{\infty}$ has the coordinates $(s,t) = (0,0)$. Also, the function x defines a separable morphism $\pi\colon X \to \mathbb{P}^1_k$ of degree 2 with $\pi(\widetilde{\infty}) = \infty$. This has ramification degree 2 at the points $P_i := (e_i, 0) \in |X|$ and at $\widetilde{\infty}$, and 1 elsewhere.

Next observe that the equation $y^2 = f(x)$ implies that $2y\,dy = f'(x)\,dx$ in $\Omega_{K/k}$, where $f' := \frac{df}{dx}$. Consider any point $P = (\xi, \eta) \in |X| \setminus \{\widetilde{\infty}\}$. In the case $\eta \neq 0$ we have $2\eta \neq 0$ and $x - \xi$ is a local uniformizer and hence $\Omega_{X/k,P} = \mathcal{O}_{X,P} \cdot dx$. By contrast, in the case $\eta = 0$ we have $f(\xi) = 0$ and hence $f'(\xi) \neq 0$, so y is a local uniformizer and $\Omega_{X/k,P} = \mathcal{O}_{X,P} \cdot dy$. At $\widetilde{\infty}$ observe that $t^2 = g(s)$ with $\mathrm{ord}_s(g) = 1$; thus s is equal to t^2 times a unit at $\widetilde{\infty}$; hence $t = y/x^2$ is a local uniformizer at $\widetilde{\infty}$ and $\Omega_{X/k,\widetilde{\infty}} = \mathcal{O}_{X,\widetilde{\infty}} \cdot dt$.

(a) Since x and y are regular functions on U, the differential $\frac{dx}{y} \in \Omega_{K/k}$ is regular on U except possibly where $y = 0$. But by the equation $2y\,dy = f'(x)\,dx$ in $\Omega_{K/k}$ it is also equal to $\frac{2dy}{f'(x)}$. In this form it extends to a regular differential at all points with $f'(x) \neq 0$; hence in particular to those with $y = 0$. At $\widetilde{\infty}$ observe that $\frac{dx}{y} = \frac{ds^{-1}}{ts^{-2}} = -\frac{ds}{t}$. But the equation $t^2 = g(s)$ implies that $2t\,dt = g'(s)\,ds$ with $g'(0) \neq 0$; hence $-\frac{ds}{t} = -\frac{2dt}{g'(s)}$ is regular at $\widetilde{\infty}$. Together this shows that $\frac{dx}{y}$ defines an element of $\Gamma(X, \Omega_{X/k})$.

We have seen in the course that every plane cubic curve has $\Omega_{X/k} \cong \mathcal{O}_X$ and hence $\dim_k \Gamma(X, \Omega_{X/k}) = 1$. It follows that the non-zero global section $\frac{dx}{y}$ generates $\Gamma(X, \Omega_{X/k})$. (*Aliter:* With the same local computations show that $\frac{dx}{y}$ is a generator of $\Omega_{X/k}$ everywhere. Thus $\Gamma(X, \Omega_{X/k}) = \Gamma(X, \mathcal{O}_X \cdot \frac{dx}{y}) = \Gamma(X, \mathcal{O}_X) \cdot \frac{dx}{y} = k \cdot \frac{dx}{y}$.)

(b) To simplify the computation of the residue at one of the points $P = (e_i, 0)$ or $\widetilde{\infty}$ where $\pi\colon X \to \mathbb{P}^1_k$ is ramified, we use the equation $\mathrm{res}_P(\omega) = \mathrm{res}_{\pi(P)}(\mathrm{tr}_{K/k(x)}(\omega))$ proved in the course, where $\mathrm{res}_{\pi(P)}$ denotes the residue taken on $\mathbb{P}^1_k$.

i. The differential dx is regular on U and hence $\mathrm{res}_P(dx) = 0$ there. At $\widetilde{\infty}$ we use the trace. Since dx comes from a differential on $\mathbb{P}^1_k$, we get $\mathrm{res}_{\widetilde{\infty}}(dx) = \mathrm{res}_\infty(\mathrm{tr}_{K/k(x)}(dx)) = \mathrm{res}_\infty(2\,dx)$. But the substitution $x = s^{-1}$ and the quotient rule show that $dx = ds^{-1} = -\frac{ds}{s^2}$ and hence $\mathrm{res}_\infty(2\,dx) = \mathrm{res}_s(-2\frac{ds}{s^2}) = 0$. Thus $\mathrm{res}_P(dx) = 0$ for all $P \in |X|$; and in particular $\sum_{P \in |X|} \mathrm{res}_P(dx) = 0$.

ii. The differential $\frac{dx}{x}$ is regular on U except where $x = 0$; hence its residues are zero there. The situation at $x = 0$ depends: If some $e_i = 0$, we have only the point $P = (0, 0)$ to consider and the morphism π is ramified there, and we get $\mathrm{res}_P(\frac{dx}{x}) = \mathrm{res}_0(\mathrm{tr}_{K/k(x)}(\frac{dx}{x})) = \mathrm{res}_x(2\frac{dx}{x}) = 2$. Otherwise there are two points $(0, \pm\eta)$ to consider with $\eta \neq 0$, and x is a uniformizer at each of them, so we directly get $\mathrm{res}_P(\frac{dx}{x}) = 1$ at these two points. At infinity we can again use the trace and the substitution $x = s^{-1}$, obtaining $\frac{dx}{x} = -\frac{ds}{s}$ and $\mathrm{res}_{\widetilde{\infty}}(\frac{dx}{x}) = \mathrm{res}_\infty(\mathrm{tr}_{K/k(x)}(\frac{dx}{x})) = \mathrm{res}_\infty(2\frac{dx}{x}) = \mathrm{res}_s(-2\frac{ds}{s}) = -2$. The sum of all residues is therefore $2 + (-2) = 0$, respectively $1 + 1 + (-2) = 0$.

iii. Since the function x and the differential $\frac{dx}{y}$ are both regular on U, so is the differential $\frac{x\,dx}{y}$. Thus its residues are zero there. At $\widetilde{\infty}$ we have $\mathrm{res}_{\widetilde{\infty}}(\frac{x\,dx}{y}) = \mathrm{res}_\infty(\mathrm{tr}_{K/k(x)}(\frac{x\,dx}{y}))$. Since $\frac{x\,dx}{y^2} = \frac{x\,dx}{f(x)}$ comes from a differential on $\mathbb{P}^1_k$, we have $\mathrm{tr}_{K/k(x)}(\frac{x\,dx}{y}) = \mathrm{tr}_{K/k(x)}(y) \cdot \frac{x\,dx}{f(x)}$. But the equation $y^2 = f(x)$ also implies that $\mathrm{tr}_{K/k(x)}(y) = 0$. Thus $\mathrm{res}_P(dx) = 0$ for all $P \in |X|$; and in particular $\sum_{P\in|X|} \mathrm{res}_P(dx) = 0$.

Aliter: To compute the residues at $\widetilde{\infty}$ without using the trace we substitute $x = s^{-1}$ and $y = ts^{-2}$ as before, so that $dx = -\frac{ds}{s^2} = \frac{-2t\,dt}{s^2 g'(s)}$ with $g'(0) \neq 0$ and the local uniformizer t. The equation $t^2 = g(s) = s + O(s^2)$ implies an expansion $s = t^2 + \ldots$ as a power series in $k[[t^2]]$. We proceed in increasing order of difficulty:

ii. Here $\frac{dx}{x} = \frac{1}{s^{-1}} \cdot \frac{-2t\,dt}{s^2 g'(s)} = \frac{-2t^2\,dt}{sg'(s)t} = -2 \cdot \frac{g(s)}{sg'(s)} \cdot \frac{dt}{t}$, where $\frac{g(s)}{sg'(s)}$ is a unit at $\widetilde{\infty}$ with constant term 1. Thus $\frac{dx}{x} = -2 \cdot \frac{dt}{t} +$ (a differential regular at $\widetilde{\infty}$); hence the residue is -2.

iii. Here $\frac{x\,dx}{y} = \frac{s^{-1}}{ts^{-2}} \cdot \frac{-2t\,dt}{s^2 g'(s)} = \frac{-2}{sg'(s)} \cdot dt$ has a pole of order 2 at $\widetilde{\infty}$. But expanding $\frac{-2}{sg'(s)}$ yields a Laurent series in t^2; hence the expansion of $\frac{-2}{sg'(s)} \cdot dt$ does not contain the term $\frac{dt}{t}$; so the residue is 0.

i. Here $dx = \frac{-2t^2\,dt}{s^2 g'(s)t} = \frac{-2g(s)}{s^2 g'(s)} \cdot \frac{dt}{t}$ has a pole of order 3 at $\widetilde{\infty}$, so we must calculate initial parts of Laurent series in order to identify the coefficient of $\frac{dt}{t}$. Write $g(s) = s - es^2 + O(s^3)$ with $e \in k$. Then the equation $t^2 = g(s)$ implies that $s = t^2 + es^2 + O(s^3) = t^2 + et^4 + O(t^6)$. Therefore

$$\frac{g(s)}{s^2 g'(s)} = \frac{s - es^2 + O(s^3)}{s^2(1 - 2es + O(s^2))} = \frac{1 + es + O(s^2)}{s} = \frac{1 + et^2 + O(t^4)}{t^2 + et^4 + O(t^6)} = \frac{1 + O(t^4)}{t^2}.$$

Thus $dx = -2 \cdot \frac{1+O(t^4)}{t^2} \cdot \frac{dt}{t} = -2 \cdot \frac{dt}{t^3}+$(a differential regular at $\widetilde{\infty}$); hence the residue is 0.

If some $e_i = 0$, we proceed in the same way to compute the residue at $P = (0, 0)$: Here $\frac{dx}{x} = \frac{2y\,dy}{f'(x)x} = \frac{2y^2\,dy}{f'(x)xy} = 2 \cdot \frac{f(x)}{f'(x)x} \cdot \frac{dy}{y}$ and $f(x) = ax + O(x^2)$ for some $a \in k^\times$; hence $\frac{f(x)}{f'(x)x}$ is a unit at P with constant term 1. Thus $\frac{dx}{x} = 2 \cdot \frac{dy}{y}+$(a differential regular at P); hence the residue is 2.