Solutions 13

Riemann-Roch, Embeddings in Projective Space

1. Let k be an arbitrary field with algebraic closure $\bar{k}$. Let X be a geometrically integral projective curve over k with $h^1(X, \mathcal{O}_X) = 0$. Show:

   (a) The base change $X_{\bar{k}}$ is isomorphic to $\mathbb{P}^1_{\bar{k}}$.

   (b) The curve X is isomorphic to a plane curve of degree 2.

   (c) We have $X \cong \mathbb{P}^1_k$ if and only if $X(k) \neq \varnothing$.

Solution: (a) We proved in the course that $H^1(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}) \cong H^1(X, \mathcal{O}_X) \otimes_k \bar{k}$ (flat base change). This implies that $X_{\bar{k}}$ has genus 0, and is hence isomorphic to $\mathbb{P}^1_{\bar{k}}$.

(b) By flat base change we also have $H^0(X_{\bar{k}}, \Omega^\vee_{X_{\bar{k}}}) \cong H^0(X, \Omega^\vee_X) \otimes_k \bar{k}$. We showed in the course that $\Omega^\vee_{X_{\bar{k}}}$ is very ample and determines the anticanonical embedding $X_{\bar{k}} \hookrightarrow \mathbb{P}^2_{\bar{k}}$ whose image is a plane curve of degree 2. By choosing global sections of $\Omega^\vee_{X_{\bar{k}}}$ that are already defined over k, this is obtained by base change from a morphism $X \to \mathbb{P}^2_k$. This morphism is non-constant and hence finite. We conclude that it is a closed embedding using the lemma below (which by the way also holds without the assumption "affine"). Since the images of $X_{\bar{k}}$ and X are defined by the same polynomial, see that X also has degree 2.

Lemma: An affine morphism $X \to Y$ of schemes over k is a closed embedding, resp. an isomorphism, if and only if the base change $X_{\bar{k}} \to Y_{\bar{k}}$ has that property.

Proof. The problem is local on Y, so it reduces to the case that Y is affine. Then $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ for some B-algebra A, and correspondingly $Y_{\bar{k}} = \operatorname{Spec} B \otimes_k \bar{k}$ and $X_{\bar{k}} = \operatorname{Spec} A \otimes_k \bar{k}$. Now $X \to Y$ is a closed embedding, resp. an isomorphism, if and only if $B \to A$ is surjective, resp. an isomorphism; and the analogue holds for $X_{\bar{k}} \to Y_{\bar{k}}$. But by mere linear algebra, using only that we have a linear map of k-vector spaces, we know that $B \to A$ is surjective, resp. an isomorphism, if and only if $B \otimes_k \bar{k} \to A \otimes_k \bar{k}$ has that property. $\square$

(c) If $X \cong \mathbb{P}^1_k$, then clearly $X(k) \neq \varnothing$ (take for instance the point $(0:1)$ in standard homogeneous coordinates). Conversely, suppose that $X(k)$ contains a point P. Then $\mathcal{O}_X(P)$ is an invertible sheaf of degree 1 on X, whose pullback to $X_{\bar{k}} \cong \mathbb{P}^1_{\bar{k}}$ still has degree 1. Thus $h^0(X, \mathcal{O}_X(P)) = h^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(P)) = 2$; hence $H^0(X, \mathcal{O}_X(P))$ contains a non-constant function f with a simple pole at P and no other pole. The associated embedding of function fields $k(f) \hookrightarrow K(X)$ then corresponds to a finite morphism $X \to \mathbb{P}^1_k$. Over $\bar{k}$ we have proved in the course that the morphism associated to such an f is an isomorphism. By the lemma above we deduce that $X \to \mathbb{P}^1_k$ itself is an isomorphism, as desired.

Aliter: By part (b), we may identify X with $V(f) \subset \mathbb{P}^2_k := \mathrm{Proj}(k[x,y,z])$ for a some non-zero $f \in k[x,y,z]$ homogeneous of degree 2. After a suitable change of coordinates, we may assume that $P := (1:0:0) \in X(k)$. The polynomial then has the form

$$f \;=\; x(a_{12}y + a_{13}z) + (a_{22}y^2 + a_{23}yz + a_{33}z^2).$$

Here a_{12} and a_{13} cannot both be 0, because otherwise $X_{\bar{k}}$ would be singular at P. After possibly interchanging y and z we may assume that $a_{13} \neq 0$. Then, after applying the linear substitution $z \rightsquigarrow a_{12}y + a_{13}z$ while keeping the other variables fixed, we may suppose that

$$f \;=\; xz + (a_{22}y^2 + a_{23}yz + a_{33}z^2) \;=\; (x + a_{23}y + a_{33}z)z + a_{22}y^2.$$

Thereafter the substitution $x \rightsquigarrow x + a_{23}y + a_{33}z$ with the other variables fixed brings the polynomial into the form

$$f \;=\; xz + a_{22}y^2.$$

Here a_{22} must be non-zero, because f is irreducible and hence not a multiple of z. The final substitution $a_{22}z \rightsquigarrow z$ thus brings the equation into the form $xz - y^2$. But we already know that $\bar{V}(xz - y^2) \subset \mathbb{P}^2_k$ is the image of the 2-uple embedding $\mathbb{P}^1_k \hookrightarrow \mathbb{P}^2_k$. Thus $X \cong \mathbb{P}^1_k$, as desired.

*2. Let X be an irreducible smooth projective curve of genus $g = 1$ over an algebraically closed field k. Show that there exists a locally free sheaf of rank 2 on X which is not a direct sum of invertible sheaves.

(For instance let i_P denote the embedding of a closed point P into X, let $\mathcal{E}$ be the kernel of a homomorphism $\mathcal{O}_X(P) \oplus \mathcal{O}_X \to i_{P*}k$ which is non-zero on each direct summand, and show that the resulting short exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}_X \xrightarrow{\binom{1}{0}} \mathcal{E} \xrightarrow{(0,1)} \mathcal{O}_X \longrightarrow 0$$

does not split.)

Solution: Define $\mathcal{E}$ as described. Then the projection to the second factor $(0,1) : \mathcal{E} \to \mathcal{O}_X$ is surjective and its kernel is the subsheaf $\mathcal{E} \cap (\mathcal{O}_X(P) \oplus 0) = \mathcal{O}_X \oplus 0$; hence the sequence $(*)$ is exact. Also, since X has genus 1, we have $H^0(\mathcal{O}_X(P)) = H^0(\mathcal{O}_X) = k$ and hence $H^0(\mathcal{O}_X(P) \oplus \mathcal{O}_X) = H^0(\mathcal{O}_X^{\oplus 2}) = k^{\oplus 2}$. Since $\mathcal{E} \cap \mathcal{O}_X^{\oplus 2} = \mathcal{O}_X \oplus \mathcal{O}_X(-P)$ and $H^0(\mathcal{O}_X(-P)) = 0$, it follows that $H^0(\mathcal{E}) = H^0(\mathcal{E} \cap \mathcal{O}_X^{\oplus 2}) = k \oplus 0$. Taking global sections the sequence $(*)$ yields a left exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(\mathcal{O}_X) & \xrightarrow{\text{(a)}} & H^0(\mathcal{E}) & \xrightarrow{\text{(b)}} & H^0(\mathcal{O}_X) \\
 & & \| & & \| & & \| \\
0 & \longrightarrow & k & \longrightarrow & k \oplus 0 & \longrightarrow & k
\end{array}$$

where (a) is an isomorphism and hence (b) is zero. But if the sequence $(*)$ were split, it would yield a short exact sequence of H^0 groups. Thus the sequence does not split.

Now suppose that $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ is a direct sum of invertible sheaves in an arbitrary way. Then each $\mathcal{L}_i$ is either a subsheaf of $\mathcal{O}_X \oplus 0$ or the composite homomorphism $\mathcal{L}_i \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{O}_X$ is non-zero. In each case $\mathcal{L}_i$ is isomorphic to a subsheaf of $\mathcal{O}_X$. In particular $\deg(\mathcal{L}_i) \leqslant \deg(\mathcal{O}_X) = 0$. But the exact sequence $(*)$ already implies that $\deg(\mathcal{L}_1) + \deg(\mathcal{L}_2) = \deg(\mathcal{E}) = \deg(\mathcal{O}_X) + \deg(\mathcal{O}_X) = 0$. Thus we must have $\deg(\mathcal{L}_i) = 0$ and hence $\mathcal{L}_i \cong \mathcal{O}_X$. This implies that $\mathcal{E} \cong \mathcal{O}_X^{\oplus 2}$ and hence $H^0(\mathcal{E}) \cong H^0(\mathcal{O}_X^{\oplus 2}) = k^{\oplus 2}$. But we have seen above that $\dim_k H^0(\mathcal{E}) = 1$, yielding a contradiction. Thus $\mathcal{E}$ is a locally free sheaf of rank 2 which is not a direct sum of invertible sheaves.

3. Let X be a smooth, irreducible curve of genus g over an algebraically closed field. Let D be an effective divisor on X. Show that:

   (a) $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D + 1$.

   (b) $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D$ if and only if $\deg(D) \geqslant 1$ and $g \geqslant 1$.

   (c) $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D - 1$ if $\deg(D) \geqslant 2$ and X is not hyperelliptic.

   Solution: We first prove the following general lemma:

   Lemma: For any divisor D_0 and any effective divisor D' we have
$$h^0(\mathcal{O}_X(D_0 + D')) \ \leqslant\ H^0(\mathcal{O}_X(D_0)) + \deg D'.$$

   Indeed, this is trivial if $D' = 0$. If $D' = P$ for a closed point P, the quotient sheaf $\mathcal{O}_X(D_0 + P)/\mathcal{O}_X(D_0)$ has stalk k at P and is zero elsewhere. Thus we have a left exact sequence $0 \to H^0(\mathcal{O}_X(D_0)) \to H^0(\mathcal{O}_X(D_0 + P)) \to k$ which implies the desired inequality. The general case follows by induction on $\deg D'$.

   Now let D be any effective divisor. With $D_0 = 0$ and $D' = D$ the lemma shows that $h^0(\mathcal{O}_X(D)) \leqslant h^0(\mathcal{O}_X) + \deg D$. Since $h^0(\mathcal{O}_X) = 1$, this proves (a).

   To prove (b) suppose first that $\deg(D) \geqslant 1$ and $g \geqslant 1$. Write $D = D_0 + D'$ with effective divisors D_0 and D' of respective degrees 1 and $\deg D - 1$. Since $g \geqslant 1$, we then have $h^0(\mathcal{O}_X(D_0)) \leqslant 1$. The lemma thus shows that $h^0(\mathcal{O}_X(D)) \leqslant h^0(\mathcal{O}_X(D_0)) + \deg D - 1 \leqslant 1 + \deg D - 1 = \deg D$. This proves the "if" part of (b). For the "only if" part suppose first that $\deg D = 0$. Since D is effective, this means that $D = 0$ and hence $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X) = 1 = \deg D + 1$. Suppose next that $g = 0$, so that without loss of generality $X = \mathbb{P}^1_k$. Then $\mathcal{O}_X(D) \cong \mathcal{O}_X(\deg D)$ with $\deg D \geqslant 0$ because D is effective; hence $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(\deg D)) = \deg D + 1$. This proves the "only if" part of (b).

   For (c) write $D = D_0 + D'$ with effective divisors D_0 and D' of respective degrees 2 and $\deg D - 2$. Since X is not hyperelliptic, we then have $h^0(\mathcal{O}_X(D_0)) \leqslant 1$. The lemma thus shows that $h^0(\mathcal{O}_X(D)) \leqslant h^0(\mathcal{O}_X(D_0)) + \deg D - 2 \leqslant 1 + \deg D - 2 = \deg D - 1$, as desired.

4. Let X be a curve of genus 2. Show that a divisor D on X is very ample if and only if $\deg D \geqslant 5$.

   Solution: In the lecture we proved that every divisor of degree $\geqslant 2g+1 = 5$ is very ample. Conversely suppose that D is very ample. Then X embeds into $\mathbb{P}^n_k$ for $n := h^0(X, \mathcal{O}(D)) - 1$. Since X is not rational, we must have $n \geqslant 2$. If $n = 2$, we have a smooth plane curve of some degree d and hence of genus $\frac{(d-1)(d-2)}{2}$, which is never 2. Therefore $n \geqslant 3$, or in other words $h^0(X, \mathcal{O}(D)) \geqslant 4$. Next observe that $\deg D > 0$ because D is ample. Since X is not rational, by Exercise 3 (b) it follows that $h^0(X, \mathcal{O}_X(D)) \leqslant \deg(D)$. Therefore $\deg D \geqslant 4$. This implies that

   $$\deg(\Omega_X \otimes \mathcal{O}_X(-D)) = 2g - 2 - \deg D \leqslant -2$$

   and hence $h^0(X, \Omega_X \otimes \mathcal{O}_X(-D)) = 0$. By Riemann-Roch, we can now deduce that

   $$4 \ \leqslant\ h^0(X, \mathcal{O}_X(D)) \ =\ \chi(X, \mathcal{O}_X(D)) \ =\ (1-2) + \deg D$$

   and hence $\deg D \geqslant 5$, as desired.

5. Let X be an irreducible smooth plane projective curve of degree 4 over an algebraically closed field k.

   (a) Show that the given embedding $X \hookrightarrow \mathbb{P}^2_k$ is the canonical embedding of X.

   (b) Deduce that X is not hyperelliptic.

   (c) Show that the effective canonical divisors on X are precisely the divisors of the form $X \cap L$ for all lines L in $\mathbb{P}^2_k$.

   Solution: (a) Observe first that the embedding is determined by three sections in $H^0(X, \mathcal{O}_X(1))$ which generate the sheaf $\mathcal{O}_X(1)$. These sections must be k-linearly independent, because any linear dependence would force X into a line in $\mathbb{P}^2_k$ and hence make its genus 0. Next, since X is a curve of degree 4, by §5.10 of the course we have $\Omega_{X/k} = \omega_{X/k} \cong \mathcal{O}_X(4-2-1) = \mathcal{O}_X(1)$. On the other hand, by §7.5 of the course the genus of X is $\frac{(4-2)(4-1)}{2} = 3$; hence $h^0(X, \Omega_{X/k}) = 3$. Thus the given embedding is determined by three k-linearly independent sections in $H^0(X, \mathcal{O}_X(1)) \cong H^0(X, \Omega_{X/k})$, which already has dimension 3; hence by a basis of $H^0(X, \Omega_{X/k})$. It is therefore the canonical embedding.

   (b) For a curve X of genus $\geqslant 1$ the canonical morphism is a closed embedding if and only if X is not hyperelliptic.

   (c) By definition a divisor D is called canonical if and only if $\mathcal{O}_X(D) \cong \Omega_{X/k}$. For any effective canonical divisor D the composite homomorphism $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D) \cong$

$\Omega_{X/k}$ amounts to a non-zero section $\omega \in H^0(X, \Omega_{X/k})$ with $\operatorname{div}(\omega) = D$. Conversely, for any non-zero section $\omega \in H^0(X, \Omega_{X/k})$, multiplication by ω yields a homomorphism $\mathcal{O}_X \hookrightarrow \Omega_{X/k}$ which extends to an isomorphism $\mathcal{O}_X(D) \cong \Omega_{X/k}$ for the effective divisor $D = \operatorname{div}(\omega)$. Thus the effective canonical divisors are precisely the divisors of the form $\operatorname{div}(\omega)$ for all non-zero $\omega \in H^0(X, \Omega_{X/k})$. Under the isomorphism $\Omega_{X/k} \cong \mathcal{O}_X(1)$ these are precisely the divisors of the form $\operatorname{div}(\ell)$ for all non-zero $\ell \in H^0(X, \mathcal{O}_X(1))$, or again for all non-zero linear polynomials in the three coordinates on $\mathbb{P}^2_k$. These are precisely the divisors associated to the finite subschemes of the form $X \cap L$ for all lines $L \subset \mathbb{P}^2_k$.