Solutions 2

Coherent and Quasi-coherent Sheaves

Convention: Let $f\colon X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. We will often refer to the direct (resp. inverse) image sheaf $f_*\mathcal{F}$ (resp. $f^*\mathcal{G}$) as the pushforward of $\mathcal{F}$ (resp. pullback of $\mathcal{G}$).

Fix a locally noetherian scheme X.

1. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals that is locally free as an $\mathcal{O}_X$-module. Show that $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module, unless … what?

   Solution: Let U be a non-empty open affine subscheme of X such that $\mathcal{I}|_U$ is free. Then $U = \operatorname{Spec} A$ and $\mathcal{I} = \tilde{I}$ for an ideal $I \subset A$ and $I \cong A^{(S)}$ as an A-module for some set S. Here $S = \varnothing$ if and only if $I = 0$, which is of course possible. If $|S| = 1$, then $\mathcal{I}|_U$ is free of rank 1, hence invertible. We claim that these are the only possible cases.

   Since A is non-zero, it possesses a minimal prime ideal $\mathfrak{p}$. Then $A_\mathfrak{p}$ is a local ring with precisely one prime ideal $\mathfrak{p}_\mathfrak{p}$. If $S \neq \varnothing$, the ideal $I_\mathfrak{p} \subset A_\mathfrak{p}$ contains an element x such that the map $A_\mathfrak{p} \to A_\mathfrak{p}x$, $y \mapsto yx$ is an isomorphism. Thus x is not a zero divisor of $A_\mathfrak{p}$; hence there exists a prime ideal $\mathfrak{q} \subset A_\mathfrak{p}$ with $x \notin \mathfrak{q}$. Then $\mathfrak{q} = \mathfrak{p}_\mathfrak{p}$; hence $x \in A_\mathfrak{p} \smallsetminus \mathfrak{p}_\mathfrak{p} = A_\mathfrak{p}^\times$. Therefore $I_\mathfrak{p} = A_\mathfrak{p}$, which is a free $A_\mathfrak{p}$-module of rank 1 as well as of rank $|S|$; hence $|S| = 1$, as desired.

   Varying U the above claim implies that X is the disjoint union of two open subschemes X_0 and X_1 such that $\mathcal{I}|_{X_0} = 0$ and $\mathcal{I}|_{X_1}$ is invertible. An appropriate answer is therefore that $\mathcal{I}$ is invertible precisely if $\mathcal{I}_x \neq 0$ for all $x \in X$.

   (If X is locally noetherian, one can also argue as follows: Since A is noetherian, the local ring $A_\mathfrak{p}$ is non-zero artinian, hence of finite positive length, say ℓ. The length of $I_\mathfrak{p}$ is thus $\leqslant \ell$, but since $I_\mathfrak{p} \cong A_\mathfrak{p}^{(S)}$, it is also $= |S| \cdot \ell$. Therefore $|S| \leqslant 1$.)

*2. For any short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, show that if two of $\mathcal{F}'$, $\mathcal{F}$, $\mathcal{F}''$ are quasi-coherent, resp. coherent, so is the third.

   Solution: See Hartshorne, Proposition II.5.7.

3. Show that if $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, so is $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

   Solution: Since the question is local, we may assume that $X = \operatorname{Spec} A$ is affine, and that $\mathcal{F} = \tilde{M}$ and $\mathcal{G} = \tilde{N}$, where M and N are finitely generated A-modules. By part (a) of Sheet 1, exercise 5, we have $\mathscr{H}om_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) \cong (\operatorname{Hom}_A(M, N))^\sim$. We are thus reduced to showing that $\operatorname{Hom}_A(M, N)$ is finitely generated as an A-module.

For this we observe that $M \cong A^n/N$ for some $n \in \mathbb{Z}^{\geqslant 0}$ and some submodule $L \subset A^n$. Thus $\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_A(A^n/L, N)$. By the left exactness of Hom the latter is an A-submodule of $\operatorname{Hom}_A(A^n, N) \cong N^n$, which is itself finitely generated. Since we assumed A is noetherian, every submodule of a finitely generated module is also finitely generated, and the desired result follows.

4. Consider a morphism $f\colon X \to Y$. Is there a natural homomorphism between $(f_*\mathcal{E}) \otimes_{\mathcal{O}_Y} (f_*\mathcal{F})$ and $f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$? When is it an isomorphism?

   Solution: There is a natural homomorphism, defined as follows: For any open $V \subset Y$ there is a natural homomorphism

$$\begin{aligned} \mathcal{E}(f^{-1}(V)) \otimes_{\mathcal{O}_Y(V)} \mathcal{F}(f^{-1}(V)) &\to \mathcal{E}(f^{-1}(V)) \otimes_{\mathcal{O}_X(f^{-1}(V))} \mathcal{F}(f^{-1}(V)) \\ s \otimes t &\mapsto s \otimes t. \end{aligned}$$

   Composing this with the sheafification morphism to $(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})(f^{-1}(V))$, we obtain a morphism of presheaves $\big((f_*\mathcal{E}) \otimes_{\mathcal{O}_Y} (f_*\mathcal{F})\big)_{pre} \to f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$, and the universal property of sheafification yields the desired morphism

$$\alpha\colon (f_*\mathcal{E}) \otimes_{\mathcal{O}_Y} (f_*\mathcal{F}) \to f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}).$$

   This is not usually an isomorphism. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ and suppose $\mathcal{E} = \mathcal{F} = \mathcal{O}_X$. The morphism f corresponds to a homomorphism $f^\flat\colon A \to B$, making B into an A-algebra. For a B-module M, we write $M|_A$ for M as an A-module. Then $(f_*\mathcal{E}) \otimes_{\mathcal{O}_Y} (f_*\mathcal{F}) \cong (B|_A \otimes_A B|_A)^\sim$ and $f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \cong ((B \otimes_B B)|_A)^\sim \cong (B|_A)^\sim$. Thus α being an isomorphism is equivalent to $B|_A \otimes_A B|_A \cong B|_A$. A typical counterexample is $\mathbb{C} \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{C} \times \mathbb{C} \not\cong \mathbb{C}$.

   We claim that α is an isomorphism if f is a closed embedding. Indeed, since the question is local, we may reduce to the case where $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} A/I$ for some ideal $I \subset A$. We may assume further that $\mathcal{E} = \tilde{M}$ and $\mathcal{F} = \tilde{N}$, for A/I-modules M and N. Then α corresponds to the natural isomorphism $M \otimes_A N \xrightarrow{\sim} M \otimes_{A/I} N$, $m \otimes n \mapsto m \otimes n$.

5. Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{G}$ and $\mathcal{G}'$ be two $\mathcal{O}_Y$-modules. Define a natural homomorphism of $\mathcal{O}_X$-modules

$$\alpha\colon f^*\mathscr{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{G}') \to \mathscr{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, f^*\mathcal{G}'),$$

   functorial in $\mathcal{G}$ and $\mathcal{G}'$. Show that α is an isomorphism if $\mathcal{G}$ is locally free of finite rank.

   Solution: By adjunction, defining α is equivalent to defining a morphism

$$\beta\colon \mathscr{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{G}') \to f_*\mathscr{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, f^*\mathcal{G}').$$

Let $V \subset Y$ be open, and let $U := f^{-1}(V)$. We have

$$f_*\mathscr{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, f^*\mathcal{G}')(V) = \mathrm{Hom}_{\mathcal{O}_U}(f^*\mathcal{G}|_U, f^*\mathcal{G}'|_U).$$

Consider the following commutative diagram:

$$\begin{array}{ccc} U & \overset{i}{\hookrightarrow} & X \\ \downarrow{\scriptstyle \overline{f}} & & {\scriptstyle f}\downarrow \\ V & \underset{j}{\hookrightarrow} & Y, \end{array}$$

where i and j are the natural inclusions and $\overline{f} := f|_U$. Then we have

$$f^*\mathcal{G}|_U = i^*(f^*\mathcal{G}) \cong (f \circ i)^*\mathcal{G} = (j \circ \overline{f})^*\mathcal{G} \cong \overline{f}^*(j^*\mathcal{G}) = \overline{f}^*(\mathcal{G}|_V)$$

and similarly for $\mathcal{G}'$. Thus there exists a natural isomorphism

$$\varphi\colon \mathrm{Hom}_{\mathcal{O}_U}(\overline{f}^*(\mathcal{G}|_V), \overline{f}^*(\mathcal{G}'|_V)) \xrightarrow{\sim} f_*\mathscr{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, f^*\mathcal{G}')(V).$$

Since $\overline{f}^*$ is a functor, there is a natural map

$$\tilde{\beta}_V\colon \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G}|_V, \mathcal{G}'|_V) \to \mathrm{Hom}_{\mathcal{O}_U}(\overline{f}^*(\mathcal{G}|_V), \overline{f}^*(\mathcal{G}'|_V)).$$

This is in fact a morphism of $\mathcal{O}_Y(V)$-modules and is compatible with restriction. We define $\beta_V := \varphi \circ \tilde{\beta}_V$. Varying V, we obtain the desired morphism of sheaves β.

We turn to the question of showing α is an isomorphism when $\mathcal{G}$ is locally free of finite rank. Since the question is local on the base, we reduce to the case where $\mathcal{G} = \mathcal{O}_Y^n$ for $n \in \mathbb{Z}^{\geqslant 0}$. The case $n = 0$ is trivial. For $n \geqslant 1$, we first use the fact that $\mathscr{H}om$, f^*, and f_* all commute with finite sums to reduce to the case where $n = 1$. In this case we have natural isomorphisms $f^*\mathscr{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{G}') \cong f^*\mathcal{G}'$ and $\mathscr{H}om_{\mathcal{O}_X}(f^*\mathcal{O}_Y, f^*\mathcal{G}') \cong f^*\mathcal{G}'$, where we have used Sheet 1, exercise 2b and the fact that $f^*\mathcal{O}_Y \cong \mathcal{O}_X$. So we at least know that both sides are isomorphic. To see that α is itself an isomorphism, we use similar identifications to show that β corresponds to the adjunction $\mathcal{G}' \to f_*f^*\mathcal{G}'$. It follows that α corresponds to the identity on $f^*\mathcal{G}'$ (by definition of the adjunction morphism) and is thus an isomorphism.

Note: Showing α is an isomorphism is **not** equivalent to showing β is an isomorphism.

6. Prove that for any noetherian scheme X:

   (a) Suppose X is affine. Prove that any quasi-coherent sheaf on X is a sum of coherent subsheaves.

   (b) For general X, show that any coherent sheaf on an open subscheme of X is the restriction of a coherent sheaf on X.

**(c) Prove (a) for general X.

Hint: For (b), first prove the affine case. For the embedding $j : U \hookrightarrow X$ and a coherent sheaf $\mathcal{F}$ on U look at the sheaf $j_*\mathcal{F}$.

Solution:

(a) When $X = \operatorname{Spec} A$, then the statement translates to the following fact: An A-module M is the sum of its finitely generated submodules. This is clear since any $m \in M$ is contained in the submodule $A \cdot m \subset M$. We return to the case of general X after proving part (b).

(b) Suppose $X = \operatorname{Spec} A$. Let $U \subset X$ be open and let $\mathcal{F}$ be a coherent sheaf on U. Since U is noetherian, we may apply Proposition II.5.8 in Hartshorne or Problem 3 of Sheet 3 to deduce that $j_*\mathcal{F}$ is quasi-coherent on X. By part (a), we have that $j_*\mathcal{F}$ is the sum of its coherent subsheaves, i.e. $j_*\mathcal{F} = \sum_{i \in I} \mathcal{G}_i$, where $\mathcal{G}_i$ runs over all coherent subsheaves of $j_*\mathcal{F}$. We note that j^* is exact in this case since it corresponds to the exact functor $\mathcal{G} \mapsto \mathcal{G}|_U$. Thus j^* preserves inclusion, so the $j^*\mathcal{G}_i$ are subsheaves of $j^*j_*\mathcal{F}$. The sum $\sum_{i \in I} \mathcal{G}_i$ is defined to be the image of the morphism $\bigoplus \mathcal{G}_i \to j_*\mathcal{F}$. Applying j^* and noting that it commutes with direct sums, we see that $j^*j_*\mathcal{F} = \sum_{i \in I} j^*\mathcal{G}_i$. Since the pullback of a coherent sheaf via a morphism of noetherian schemes is again coherent, the $j^*\mathcal{G}_i$ are coherent on U. We have thus written $j^*j_*\mathcal{F}$ as the sum of coherent subsheaves $j^*\mathcal{G}_i$.

Since j is an open embedding the adjunction $j^*j_*\mathcal{F} \to \mathcal{F}$ is an isomorphism. In particular, the sheaf $j^*j_*\mathcal{F}$ is coherent on U. Let $U = \cup_{k=1}^n (U_k := \operatorname{Spec} A_k)$ be a finite affine open covering of U. Taking sections over U_k, the equality $j^*j_*\mathcal{F} = \sum_{i \in I} j^*\mathcal{G}_i$ yields an equality $M_k = \sum_{i \in I} M_{ik}$ where M_k is a finitely generated A_k-module and the $M_{ik} \subset M_k$ are submodules. Any generator of M_k is contained in some finite sum of the M_{ik}. Since the $\mathcal{G}_i$ include all coherent subsheaves of $j_*\mathcal{F}$, it follows that any finite sum of the $\mathcal{G}_i$ is equal to $\mathcal{G}_j$ for some $j \in I$. Putting everything together and using that M_k is finitely generated, we find that $M_k = M_{ik}$ for some $i \in I$. Since there are finitely many U_k, we may choose an i that works for all of them. This implies that $j^*j_*\mathcal{F} = j^*\mathcal{G}_i$. From the adjunction we obtain $j^*\mathcal{G}_i \cong \mathcal{F}$. Thus proves (b) in the affine case.

Now let X be an arbitrary noetherian scheme. Cover X with finitely many affine $U_1, \ldots, U_n$. Let $\mathcal{F}_1 := \mathcal{F}|_{U \cap U_1}$. By the affine case, we know that $\mathcal{F}_1$ extends to a coherent sheaf $\mathcal{F}_1'$ on U_1. Since $\mathcal{F}_1'|_{U \cap U_1} = \mathcal{F}|_{U \cap U_1}$, we may glue $\mathcal{F}$ and $\mathcal{F}_1'$ to obtain a coherent sheaf $\mathcal{F}'$ over $U \cup U_1$ extending $\mathcal{F}$. We may repeat this argument with U_2 in place of U_1 and $(U_1 \cup U) \cap U_2$ in place of $U \cap U_1$ and $\mathcal{F}'$ in place of $\mathcal{F}$ to obtain a coherent sheaf $\mathcal{F}''$ on $U_2 \cup U_1 \cup U$ extending $\mathcal{F}$. Iterating yields the desired extension of $\mathcal{F}$ to a coherent sheaf on X.

We will not give a proof of (c). There are several questions related to this exercise that one can ask:

i. If $\mathcal{F}$ is quasicoherent on U, does there exist a quasicoherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}|_U \cong \mathcal{F}$?

ii. If $\mathcal{F}$ is coherent on U, does there exist a coherent sheaf $\mathcal{G}$ on X such that $\mathcal{G}|_U \cong \mathcal{F}$?

iii. If $\mathcal{G}$ is quasicoherent on X such that $\mathcal{F}$ is a coherent subsheaf of $\mathcal{G}|_U$, does there exist a coherent subsheaf $\mathcal{F}'$ of $\mathcal{G}$ such that $\mathcal{F} = \mathcal{F}'|_U$?

iv. Is every quasicoherent sheaf on X the sum of its coherent subsheaves?

We have just answered the first and second questions affirmatively. The third and forth are related as follows: Let X be a noetherian scheme and let $\mathcal{G}$ be a quasicoherent sheaf on X. Let $\{\mathcal{G}_i\}_{i\in I}$ denote the set of coherent subsheaves of $\mathcal{G}$. There is a natural inclusion $\iota\colon \sum_{i\in I}\mathcal{G}_i \hookrightarrow \mathcal{G}$. For each affine open $U \subset X$, consider the induced inclusion $(\sum_{i\in I}\mathcal{G}_i)|_U = \sum_{i\in I}(\mathcal{G}_i|_U) \hookrightarrow \mathcal{G}|_U$. If this is an isomorphism for every such U, then ι is as well. Since we have shown that $\mathcal{G}|_U$ is the sum of its coherent subsheaves, it suffices to show that each such subsheaf is equal to $\mathcal{G}_i|_U$ for some i. This is where (iii) is important. A priori we just know that $\mathcal{G}_i|_U$ extends to some coherent sheaf on X, but our construction doesn't yield a subsheaf of $\mathcal{G}$.

If we want to show (iii) in the affine case, we can look at $j_*\mathcal{F} \subset j_*j^*\mathcal{G}$. We know there is a coherent $\mathcal{F}' \subset j_*\mathcal{F}$ extending $\mathcal{F}$. We also have the adjunction morphism $\rho\colon \mathcal{G} \to j_*j^*\mathcal{G}$. We have $\rho^{-1}(\mathcal{F}')|_U = \mathcal{F}$ since $\rho|_U$ is an isomorphism. A similar gluing as in part (b) may resolve the general case.