D-MATH
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Algebraic Geometry II

HS 2017

Solutions 3

Functoriality, Quasi-coherent Sheaves on Proj(R)

1. (*Tensor Operations on Sheaves*) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and $n \in \mathbb{Z}^{\geqslant 0}$. We define the *$n$-th exterior power* $\bigwedge^n \mathcal{F}$ (resp. *n-th symmetric power* $\mathrm{Sym}^n \mathcal{F}$) of $\mathcal{F}$ by taking the sheaf associated to the presheaf which to each open set U assigns the $\mathcal{O}_X(U)$-module $\bigwedge^n_{\mathcal{O}_X(U)} \mathcal{F}(U)$ $\big($resp. $\mathrm{Sym}^n_{\mathcal{O}_X(U)} \mathcal{F}(U)\big)$.

   (a) Suppose that if $\mathcal{F}$ is locally free of rank r, then $\bigwedge^n \mathcal{F}$ and $\mathrm{Sym}^n \mathcal{F}$ are locally free of ranks $\binom{r}{n}$ and $\binom{r+n-1}{n}$ respectively.

   (b) With $\mathcal{F}$ as in (a) show that for each $n = 0, \ldots, r$ the multiplication map $\bigwedge^n \mathcal{F} \otimes \bigwedge^{r-n} \mathcal{F} \to \bigwedge^r \mathcal{F}$ is a perfect pairing, i.e., it induces an isomorphism $\bigwedge^n \mathcal{F} \cong (\bigwedge^{r-n} \mathcal{F})^\vee \otimes \bigwedge^r \mathcal{F}$.

   (c) Let $f\colon X \to Y$ be a morphism of schemes. Show that f^* commutes with $\bigwedge^n$ and Sym^n.

Solution:

(a) Since the question is local, we may assume that $\mathcal{F}$ is free of rank r. Let $s_1, s_2, \ldots, s_r \in \mathcal{F}(X)$ be a basis, so that $\mathcal{F}(U) \cong \mathcal{O}_X(U)s_1|_U \oplus \cdots \oplus \mathcal{O}_X(U)s_r|_U$ for all open subsets $U \subset X$. Let

$$I_n := \{\underline{i} = (i_1, \ldots, i_n) \mid 1 \leqslant i_1 < \ldots < i_n \leqslant r\}$$

and abbreviate $s_{\underline{i}} := s_{i_1} \wedge \cdots \wedge s_{i_n}$ for all $\underline{i} \in I_n$. Then for any open $U \subset X$ the n-th exterior power of $\mathcal{F}(U)$ over $\mathcal{O}_X(U)$ is a free module with basis $\{s_{\underline{i}} \mid \underline{i} \in I_n\}$. As this is compatible with the restriction maps, the corresponding presheaf is already a free $\mathcal{O}_X$-module of rank $|I_n| = \binom{r}{n}$. In particular it is already a sheaf (whereas it might not be one if $\mathcal{F}$ were not already free). This proves the desired result for $\bigwedge^n \mathcal{F}$. The proof for $\mathrm{Sym}^n \mathcal{F}$ is completely analogous.

(b) First we note that the tensor product of $\mathcal{O}_X$-modules satisfies a similar universal property as the tensor product of modules over a ring. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be $\mathcal{O}_X$-modules. We call a morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$ *bilinear* if the induced morphism at every stalk is bilinear. Giving a morphism of $\mathcal{O}_X$-modules $\mathcal{F} \otimes \mathcal{G} \to \mathcal{H}$ is equivalent to giving a bilinear morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$. To prove this, one works over open subsets of X and uses the corresponding universal property for modules to show that a bilinear morphism factors uniquely through the tensor presheaf and then applies the universal property of the sheafification.

We may thus define the multiplication map $\mu\colon \bigwedge^n \mathcal{F} \otimes \bigwedge^{r-n} \mathcal{F} \to \bigwedge^r \mathcal{F}$ as follows: First we have a morphism of presheaves given on each open $U \subset X$ via

$$\bigwedge_{\mathcal{O}_X(U)}^{n} \mathcal{F}(U) \times \bigwedge_{\mathcal{O}_X(U)}^{r-n} \mathcal{F}(U) \to \bigwedge_{\mathcal{O}_X(U)}^{r} \mathcal{F}(U), \ (s,t) \mapsto s \wedge t.$$

This induces a bilinear morphism of sheaves $\bigwedge^n \mathcal{F} \times \bigwedge^{r-n} \mathcal{F} \to \bigwedge^r \mathcal{F}$, and we obtain μ from the universal property.

Since $\mathcal{F}$ is locally free, we may apply part (a) and exercise 4b on Sheet 1 to obtain a natural isomorphism $(\bigwedge^{r-n} \mathcal{F})^\vee \otimes \bigwedge^r \mathcal{F} \cong \mathscr{H}om_{\mathcal{O}_X}(\bigwedge^{r-n} \mathcal{F}, \bigwedge^r \mathcal{F})$. We claim that μ induces a morphism

$$\alpha\colon \bigwedge^{n} \mathcal{F} \to \mathscr{H}om_{\mathcal{O}_X}(\bigwedge^{r-n} \mathcal{F}, \bigwedge^{r} \mathcal{F}),$$

and that α is an isomorphism. Let $U \subset X$ be open. A section $s \in (\bigwedge^n \mathcal{F})(U)$ induces a morphism $(\bigwedge^{r-n} \mathcal{F})|_U \to (\bigwedge^r \mathcal{F})|_U$ of sheaves as follows: For each open $V \subset U$, we map $t \in (\bigwedge^{r-n} \mathcal{F})(V)$ to $\mu(s|_V \otimes t)$. Varying U, this defines the desired morphism α.

To show that α is an isomorphism, we may assume without loss of generality that $\mathcal{F}$ is free of rank r. Then by part (a), we obtain an isomorphism $\bigwedge^r \mathcal{F} \cong \mathcal{O}_X$, corresponding to a choice of basis $\mathcal{B} := \{s_1, \dots, s_r\}$ of $\mathcal{F}$. This allows us to identify the codomain of α with $(\bigwedge^{r-n} \mathcal{F})^\vee$. Let I and J denote the indexing sets of the basis elements obtained from $\mathcal{B}$ of $\bigwedge^n \mathcal{F}$ and $\bigwedge^{r-n} \mathcal{F}$ respectively. By the same calculation as in the calculus of differential forms, the basis $\{s_{\overline{i}}\}_{\overline{i} \in I}$ of $\bigwedge^n \mathcal{F}$ is dual to the basis $\{s_{\overline{j}}\}_{\overline{j} \in J}$ of $\bigwedge^{r-n} \mathcal{F}$, up to sign for the pairing.

(c) As with tensor products, there is a natural morphism $\alpha\colon \bigwedge^n f^*\mathcal{F} \to f^* \bigwedge^n \mathcal{F}$. Let M be a free module over a ring R. Consider a ring homomorphism $R \to S$ and the resulting S-module $N := S \otimes_R M$. There is a natural isomorphism $\bigwedge_S^n N \cong S \otimes_R \bigwedge_R^n M$. Let $x \in X$ and $y := f(x)$. The morphism on stalks

$$\alpha_x\colon \bigwedge_{\mathcal{O}_{X,x}}^{n} (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{F}_y) \to \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \bigwedge_{\mathcal{O}_{Y,y}}^{n} \mathcal{F}_y$$

is precisely this isomorphism. Thus α is an isomorphism.

Variant: The isomorphism $\bigwedge_S^n N \cong S \otimes_R \bigwedge_R^n M$ implies that formation of the exterior power commutes with localization. We may thus construct α locally over affine opens in X mapping to affine opens in Y under f and then glue.

2. Let $f\colon X \to Y$ be a morphism of schemes. Recall that if $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{E}$ is an $\mathcal{O}_Y$-module, then there is a natural homomorphism $f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \to f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E})$. Show that this is an isomorphism when $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank.

Solution: Without loss of generality, we may assume that $\mathcal{E} \cong \mathcal{O}_Y^n$ for some $n \in \mathbb{Z}^{\geqslant 0}$. Then there isomorphisms

$$\begin{aligned} f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} &\cong (f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y)^n \\ &\cong (f_*\mathcal{F})^n \\ &\cong f_*(\mathcal{F}^n) \\ &\cong f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^n) \\ &\cong f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y^n) \\ &\cong f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}). \end{aligned}$$

Here we have used that $\otimes$ and f^* and f_* all commute with finite direct sums, that $f^*\mathcal{O}_Y \cong \mathcal{O}_X$ and that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}$. One checks that this is exactly the natural homomorphism $f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \to f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E})$ defined in the course, which is thus an isomorphism.

*3. Let $f\colon X \to Y$ be a quasicompact, separated morphism of schemes, and suppose $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. Show that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

Solution: See Görtz and Wedhorn, Proposition 10.10. Note that it is actually enough to assume that the morphism is quasi-separated. See Görtz and Wedhorn, Proposition 10.27. In particular, the same conclusion holds for $f\colon X \to Y$ such that X is noetherian.

4. Let R be a ring. Consider a graded ideal $\mathfrak{a} \subset S := R[X_0, \ldots, X_n]$ and let $X := \operatorname{Proj} S/\mathfrak{a}$ and let $i\colon X \hookrightarrow \mathbb{P}_R^n$ be the associated closed embedding. Show that for any $k \in \mathbb{Z}$ there is a natural isomorphism $i^*\mathcal{O}_{\mathbb{P}_R^n}(k) \cong \mathcal{O}_X(k)$.

Solution: Fix some X_j and let $\overline{X}_j$ denote the image of X_j under the canonical projection $S \twoheadrightarrow S/\mathfrak{a}$. Then i induces a closed embedding

$$D_{\overline{X}_j} \cong \operatorname{Spec}(S/\mathfrak{a})_{\overline{X}_j,0} \hookrightarrow D_{X_j} \cong \operatorname{Spec} S_{X_j,0}.$$

Now we know that $\mathcal{O}_{\mathbb{P}^n}(k)(\overline{D}_{X_j}) = S(k)_{X_j,0} = S_{X_j,0} \cdot X_j^k$. Since for affine schemes pullback corresponds to tensor product, we obtain

$$i^*\mathcal{O}_{\mathbb{P}^n}(k)(D_{\overline{X}_j}) = (S_{X_j,0} \cdot X_j^k) \otimes_{S_{X_j,0}} (S/\mathfrak{a})_{\overline{X}_j,0} \cong (S/\mathfrak{a})_{\overline{X}_j,0} \cdot \overline{X}_j^k.$$

By the same reasoning as before, the left hand side is equal to $\mathcal{O}_X(k)(D_{\overline{X}_j})$. Varying j, we thus obtain a family of natural isomorphisms

$$\varphi_j\colon i^*\mathcal{O}_{\mathbb{P}^n}(k)|_{D_{\overline{X}_j}} \xrightarrow{\sim} \mathcal{O}_X(k)|_{D_{\overline{X}_j}}.$$

Since the gluing morphisms on the intersection $D_{\overline{X}_j} \cap D_{\overline{X}_\ell}$ are given by multiplication by $\overline{X}_\ell^k/\overline{X}_j^k$, we see that the φ_j are compatible with gluing and thus yield the desired isomorphism.

5. For graded modules over a graded ring R: Is the functor $M \mapsto \tilde{M}$ faithful? Full? Does it reflect isomorphisms?

   Solution: The answer to all three questions is no.

   It is a general fact that if $\alpha\colon M \to M'$ is a morphism of graded R-modules such that there exists a $d_0 \in \mathbb{Z}$ with $\alpha_d\colon M_d \to M'_d$ an isomorphism for all $d \geqslant d_0$, then the morphism $\tilde{\alpha}\colon \tilde{M} \to \tilde{M}'$ induced by α is an isomorphism. To see this, let $f \in R_+$ be homogenous. We have $\tilde{N}(D_f) \cong N_{f,0}$ and $\tilde{M}'(D_f) \cong M'_{f,0}$. The morphism α induces a morphism $\alpha_{f,0}\colon M_{f,0} \to M'_{f,0}$. Suppose $\alpha_{0,f}(m/f^n) = 0$. Multiplying the numerator and denominator for some power of f, we may assume that $\deg m \geqslant d_0$. We have $\alpha_{f,0}(m/f^n) = \alpha(m)/f^n = 0$, which means that $f^k\alpha(m) = \alpha(f^k m) = 0$ for some $k \in \mathbb{Z}^{\geqslant 0}$. But since $\deg f^k m \geqslant d_0$, this means that $f^k m = 0$, so $m/f^n = 0$ in $M_{f,0}$. Thus $\alpha_{f,0}$ is injective. A similar argument shows surjectivity. Since f was arbitrary and $\alpha_{f,0}$ determines the restriction of $\tilde{\alpha}$ to D_f, we find that $\tilde{\alpha}$ is an ismorphism over every member of an open covering, and is thus itself and isomorphism.

   Let k be a field, let $R := k[X_0, \ldots, X_n]$ and let $X := \operatorname{Proj} R$. Consider the graded R-modules $M := R$ and $M' = R_+$. Then $M \ncong M'$ since their degree 0 parts are not isomorphic. By the preceeding paragraph, the inclusion $M' \hookrightarrow M$ induces an isomorphism of the corresponding quasicoherent sheaves. This shows that the functor does not reflect isomorphisms. Moreover, the group of graded homomorphisms $\operatorname{Hom}_R(M, M') = 0$ since any such morphism sends M_0 to 0. Thus the functor is not full. Now consider the trivially graded R-module $M'' := M/M' \cong k$. The above argument shows that $\tilde{M}'' = 0$. In particular, we have $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}'', \tilde{M}) = 0$. The inclusion $\iota\colon k \hookrightarrow M$ yields a non-zero element of $\operatorname{Hom}_R(M'', M)$. It follows that the functor is not faithful.

6. Let $i\colon \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$ be the Segre embedding. Prove that $i^*\mathcal{O}(1) \cong \mathrm{pr}_1^*\mathcal{O}(1) \otimes \mathrm{pr}_2^*\mathcal{O}(1)$. What is $i^*\mathcal{O}(1)$ for the d-uple embedding $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$?

   Solution: Recall that $\mathcal{O}(1)$ on $\mathbb{P}^n$ is generated by its global sections $X_0, \ldots, X_n$. Likewise $\mathcal{O}(1)$ on $\mathbb{P}^m$ is generated by its global sections $Y_0, \ldots, Y_m$. Thus the sheaf $\mathrm{pr}_1^*\mathcal{O}(1) \otimes \mathrm{pr}_2^*\mathcal{O}(1)$ on $\mathbb{P}^n \times \mathbb{P}^m$ is generated by the global sections $X_i \otimes Y_j$ for all i and j. A quick local calculation shows that the associated morphism $i\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ is precisely the Segre embedding. Thus $i^*\mathcal{O}(1) \cong \mathrm{pr}_1^*\mathcal{O}(1) \otimes \mathrm{pr}_2^*\mathcal{O}(1)$, and we are done.

   Likewise the sheaf $\mathcal{O}(d)$ on $\mathbb{P}^n$ is generated by the global sections $X_0^{i_0} \cdots X_n^{i_n}$ for all multiindices $\geqslant 0$ with sum d, and the associated morphism $i\colon \mathbb{P}^n \to \mathbb{P}^N$ is the d-uple embedding; hence $i^*\mathcal{O}(1) \cong \mathcal{O}(d)$.

   Variant: Write $R := \mathbb{Z}[X_0, \ldots, X_n]$ and $S := \mathbb{Z}[Y_0, \ldots, Y_m]$ and $Q := \mathbb{Z}[Z_{00}, \ldots, Z_{nm}]$ and recall that the Segre embedding is obtained from the substitution $Z_{ij} = X_i Y_j$. Thus for any i and j we have $i^{-1}(D_{Z_{ij}}) = U_{ij} := D_{X_i} \times D_{Y_j}$. Since $\mathcal{O}(1)|_{D_{X_i}} = \mathcal{O} \cdot X_i$

and $\mathcal{O}(1)|_{D_{Y_j}} = \mathcal{O} \cdot Y_j$ and $\mathcal{O}(1)|_{D_{Z_{ij}}} = \mathcal{O} \cdot Z_{ij}$, we find isomorphisms

$$i^*\mathcal{O}(1)|_{U_{ij}} \;\cong\; \mathcal{O}|_{U_{ij}} \cdot Z_{ij} \;\cong\; \mathcal{O}|_{U_{ij}} \cdot X_i Y_j \;\cong\; (\mathrm{pr}_1^*\mathcal{O}(1) \otimes \mathrm{pr}_2^*\mathcal{O}(1))|_{U_{ij}}.$$

A quick calculation shows that these are compatible over all intersections $U_{ij} \cap U_{i'j'}$; hence they glue to an isomorphism $i^*\mathcal{O}(1) \cong \mathrm{pr}_1^*\mathcal{O}(1) \otimes \mathrm{pr}_2^*\mathcal{O}(1)$.