Solutions 4

Invertible Sheaves, Morphisms to Projective Space

1. Let $X = \operatorname{Proj} R$ for a graded ring R. Consider the graded ring $R' := \bigoplus_{d \geqslant 0} \mathcal{O}_X(d)(X)$. Is there a natural isomorphism $\operatorname{Proj} R' \cong X$?

   Solution: In order to have the invertible sheaf $\mathcal{O}_X(1)$ we must assume that R is generated by R_1 over R_0. We assume further that R is finitely generated over R_0.

   Lemma. *Consider any $g \in \mathcal{O}_X(d)(X) \subset R'$ for some $d > 0$. Let D^X_g be the open subscheme of X where g generates $\mathcal{O}_X(d)$. Then:*

   (a) D^X_g is affine.

   (b) There is a natural isomorphism $R'_{g,0} \xrightarrow{\sim} \mathcal{O}_X(D^X_g)$.

   Proof. Abbreviate $U := D^X_g$.

   (a) Let $f_1, \ldots, f_n \in R_1$ be generators of R over R_0. Then each $V_i := D^X_{f_i}$ is open affine and $X = \bigcup_i V_i$. Also, for each i we have

   $$D^U_{\frac{f_i^d}{g}} = U \cap V_i = D^{V_i}_{\frac{g}{f_i^d}}.$$

   The last term is a standard open subset of an affine scheme and hence affine. Thus $U = \bigcup_i (U \cap V_i) = \bigcup_i D^U_{f_i^d/g}$ is a union of open affines. Since these cover U, the elements f_i^d/g generate the unit ideal in $\mathcal{O}_U(U)$. Traveling to the future, we apply Exercise 5a on Sheet 5 to deduce that U is affine, as desired.

   (b) The homomorphism is simply defined via $\frac{h}{g^n} \mapsto \frac{h|_U}{g^n|_U}$. Suppose $\frac{h}{g^n} \mapsto 0$. We have $h \in \mathcal{O}_X(dn)(X)$, and our assumption implies that $h|_U = 0$. By part (a) of a lemma from §5.7, there exists an $m \geqslant 0$ such that $h \otimes g^m = 0$ in $\mathcal{O}_X(d(n+m))(X)$. Thus $\frac{h}{g^n} = \frac{hg^m}{g^{n+m}} = 0$ in $R'_{g,0}$. The morphism is thus injective. A similar application of part (b) of the same lemma gives surjectivity. □

   Let $X' := \operatorname{Proj} R'$. The lemma yields a natural isomorphism $D^X_g \xrightarrow{\sim} D^{X'}_g$. Since these cover X and X' respectively, the isomorphisms glue to an isomorphism $X \xrightarrow{\sim} X'$.

2. Let $X = \operatorname{Proj} R$ for a graded ring R that is generated by finitely many elements of R_1 over a noetherian ring R_0. Prove that a sheaf of $\mathcal{O}_X$-modules is coherent if and only if it is isomorphic to $\tilde{M}$ for a finitely generated graded R-module M.

Solution:

$\Rightarrow$: Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let $M := \bigoplus_{n\geqslant 0}\mathcal{F}(n)$. Then $\mathcal{F} \cong \tilde{M}$. Recall that X admits a closed embedding into $\mathbb{P}^n$ with $\mathcal{O}_X(1)$ being the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$, and hence $\mathcal{O}_X(1)$ is very ample. It is thus ample. It follows that there exists an n such that $\mathcal{F}(n)$ is generated by global sections. Since X is noetherian and $\mathcal{F}(n)$ is finitely generated, it is generated by finitely many global sections. Let M' be the graded R-submodule generated by these sections. The inclusion $M' \subset M$ induces an inclusion of sheaves $\tilde{M}' \hookrightarrow \tilde{M}$. Twisting by n this yields isomorphisms $\tilde{M}'(n) \cong \tilde{M}(n) \cong \mathcal{F}(n)$ since $\mathcal{F}(n)$ is generated by global sections in M'. Twisting by $-n$, we find $\tilde{M}' \cong \mathcal{F}$, as desired.

$\Leftarrow$: Let M be a finitely generated graded R-module, and let $f_0, \dots f_n \in R_1$ be generators of R over R_0. For each i, we have $\tilde{M}(D_{f_i}) = M_{f_i,0}$ and one checks that M being finitely generated implies the same for $M_{f_i,0}$. Since the D_{f_i} form an affine open covering and X is noetherian, it follows that $\tilde{M}$ is coherent.

3. Let $\mathcal{L}$ and $\mathcal{L}'$ be invertible sheaves on a noetherian scheme X. Show:

   (a) If $\mathcal{L}$ is ample, there exists an $n_0 \in \mathbb{Z}$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is very ample for all $n \geqslant n_0$.

   (b) If $\mathcal{L}$ is ample and there exists an integer $n' > 0$ such that $\mathcal{L}'^{\otimes n'}$ is generated by its global sections, then $\mathcal{L} \otimes \mathcal{L}'$ is ample.

   (c) If $\mathcal{L}$ and $\mathcal{L}'$ are ample, then $\mathcal{L} \otimes \mathcal{L}'$ is ample.

   Solution:

   (a) In the lecture we proved that some power $\mathcal{L}^{\otimes m}$ is very ample. In particular, $\mathcal{L}^{\otimes m}$ is generated by global sections. By the definition of ampleness applied to $\mathcal{F} := \mathcal{L}'$, there exists an n_1 such that for all $n \geqslant n_1$ the sheaf $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is generated by global sections. By a proposition in the lecture the sheaf $\mathcal{L}^{\otimes m} \otimes (\mathcal{L}^{\otimes n} \otimes \mathcal{L}') = \mathcal{L}^{\otimes m+n} \otimes \mathcal{L}'$ is then very ample. So the desired statement holds with $n_0 = m + n_1$.

   (b) See [Görtz and Wedhorn, Proposition 13.50,(3)].

   (c) By applying the definition of ample to $\mathcal{F} := \mathcal{O}_X$, there exists an integer $n' > 0$ such that $\mathcal{L}'^{\otimes n'}$ is generated by global sections. Apply (b).

4. Let $f\colon X \to Y$ be a finite morphism of noetherian schemes. Let $\mathcal{L}$ be an ample invertible sheaf on Y. Show that $f^*\mathcal{L}$ is ample.

   Solution: We give a proof that does not require the noetherian assumption. Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on X. Since f is finite and hence affine, a proposition from the course shows that $f_*\mathcal{F}$ is quasicoherent. Recall the following fact from commutative algebra: Let A be a ring and B an A-algebra, which is finitely generated as a B-module. If M is a finitely generated B-module, then $M|_A$ is a finitely generated A-module. Since f is finite, one applies this fact to deduce that $f_*\mathcal{F}$ is also finitely generated. Thus there exists an $n_0 > 0$ such that $f_*\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for all $n \geqslant n_0$. Applying f^* we find that $f^*(f_*\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \cong f^* f_* \mathcal{F} \otimes (f^*\mathcal{L})^{\otimes n}$ is generated by global sections. This is equivalent to giving a surjective morphism $\mathcal{O}_X^{(I)} \twoheadrightarrow f^* f_* \mathcal{F} \otimes (f^*\mathcal{L})^{\otimes n}$. Since f is affine, the adjunction morphism $f^* f_* \mathcal{F} \to \mathcal{F}$ is surjective. This corresponds to the fact that the natural morphism $M|_A \otimes_A B \to M$ is surjective. Tensoring with $(f^*\mathcal{L})^{\otimes n}$ is left exact. The composition $\mathcal{O}_X^{(I)} \to f^* f_* \mathcal{F} \otimes (f^*\mathcal{L})^{\otimes n} \to \mathcal{F} \otimes (f^*\mathcal{L})^{\otimes n}$ is therefore also surjective. Hence $\mathcal{F} \otimes (f^*\mathcal{L})^{\otimes n}$ is generated by global sections and $f^*\mathcal{L}$ is ample.

5. Let X be the affine line over a field k with the origin doubled. Calculate $\operatorname{Pic} X$, determine which invertible sheaves are generated by global sections, and then show that there is no ample invertible sheaf on X.

   Solution: We first show that $\operatorname{Pic}(\mathbb{A}^1_k) = 0$. Let $\mathcal{L}$ be an invertible sheaf on $\mathbb{A}^1_k$. In particular, it is coherent and thus $\mathcal{L} \cong \tilde{M}$, where M is a finitely generated $k[t]$-module which is locally free of rank 1. This implies that M is torsion free. Since $k[t]$ is a principal ideal domain, it follows that M is free of rank 1. Thus $\mathcal{L}$ is trivial, as desired. Alternatively, since $k[t]$ is a Dedekind domain, there is an isomorphism $\operatorname{Pic}(\mathbb{A}^1) \cong \operatorname{Cl}(k[t])$. The latter is trivial since $k[t]$ is a factorial ring.

   Let U and U' be the two copies of $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ in X. To give an invertible sheaf on X is equivalent to giving invertible sheaves $\mathcal{L}$ on U and $\mathcal{L}'$ on U' along with a gluing isomorphism $\mathcal{L}|_{U\cap U'} \xrightarrow{\sim} \mathcal{L}'|_{U \cap U'}$. Since $\operatorname{Pic}(\mathbb{A}^1_k) = 0$, we may assume $\mathcal{L} = \mathcal{L}' = \mathcal{O}_{\mathbb{A}^1_k}$. A gluing isomorphism then corresponds to multiplication by an element of $\mathcal{O}_X^\times(U \cap U') = k[t, t^{-1}]^\times = k^\times \cdot t^{\mathbb{Z}}$. Fix $n \in \mathbb{Z}$ and $\mu, \lambda \in k^\times$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{A}^1_k}|_{U\cap U'} & \xrightarrow[\sim]{\cdot \lambda t^n} & \mathcal{O}_{\mathbb{A}^1_k}|_{U\cap U'} \\ \| & & \downarrow{\scriptstyle \cdot\lambda^{-1}\mu} \\ \mathcal{O}_{\mathbb{A}^1_k}|_{U\cap U'} & \xrightarrow[\sim]{\cdot \mu t^n} & \mathcal{O}_{\mathbb{A}^1_k}|_{U\cap U'}. \end{array}$$

   Since multiplication by $\lambda^{-1}\mu$ is an automorphism of $\mathcal{O}_{\mathbb{A}^1_k}$, it follows that the invertible sheaves obtained from gluing via λt^n and μt^n are isomorphic. Thus every invertible sheaf on X is isomorphic to the invertible sheaf $\mathcal{L}_n$ obtained by gluing via multiplication by t^n for some $n \in \mathbb{Z}$.

   Suppose $n \neq m \in \mathbb{Z}$. We have $\mathcal{L}_n(X) \cong \{(f, g) \in k[x]^2 \mid t^n f = g\}$. Any isomorphism $\mathcal{L}_n \to \mathcal{L}_m$ would induce an isomorphism over U and U', which corresponds to multiplication by a pair $(\lambda, \mu) \in (k^\times)^2$. In particular, on global sections we would have $(f, g) \mapsto (\lambda f, \mu g)$. This means $t^n f = g$ and $t^m \lambda f = \mu g$. Comparing degrees, this yields a contradiction.

   We thus have a bijection of sets $\mathbb{Z} \to \operatorname{Pic}(X)$, $n \mapsto [\mathcal{L}_n]$. It remains to show that this is a group homomorphism i.e. that $\mathcal{L}_n \otimes \mathcal{L}_m \cong \mathcal{L}_{n+m}$. Since restriction commutes with tensor product, the sheaf $\mathcal{L}_n \otimes \mathcal{L}_m$ is obtained by gluing two copies of $\mathcal{O}_{\mathbb{A}^1_k} \otimes_{\mathcal{O}_{\mathbb{A}^1_k}} \mathcal{O}_{\mathbb{A}^1_k}$ via multiplication by $t^n \otimes t^m$. Contracting the tensor product yields the desired isomorphism.

Now suppose X admits an ample invertible sheaf $\mathcal{L}$. Since X is noetherian, this means that some power of $\mathcal{L}$ is very ample, so X admits a locally closed embedding into some $\mathbb{P}^n_k$. But this would imply that X is separated, a contradiction. Alternatively, one can show that the global sections of $\mathcal{L}_n$ do not generate it when $n \neq 0$. This implies that none of the $\mathcal{L}_n$ for $n \neq 0$ are ample, since some power of a given ample invertible sheaf is always generated by global sections. It also implies that $\mathcal{O}_X$ is not ample, since then every finitely generated quasi-coherent sheaf on X would be generated by global sections.