Solutions 5

Projective Morphisms, Invertible Sheaves, Relative Proj

1. Prove that a morphism $f\colon X \to Y$ is projective if and only if it is quasiprojective and proper.

   Solution: "$\Rightarrow$": That projective implies quasiprojective is clear. By a proposition of §4.12, projective morphisms are proper.

   "$\Leftarrow$": Let $Y = \bigcup_i V_i$ be an open covering such that for each i there is a locally closed embedding $e_i : f^{-1}(V_i) \hookrightarrow \mathbb{P}^{n_i} \times V_i$. Then $\text{proj}_i \circ e_i = f|_{f^{-1}(V_i)}$ is proper and proj_i is separated; hence e_i is proper by a proposition of §4.12. Thus e_i is a closed embedding; hence f is projective.

2. Let $f\colon X \to Y$ be a quasicompact morphism. Let $\mathcal{L}$ and $\mathcal{L}'$ be invertible sheaves on X that are relatively ample over Y. Show that $\mathcal{L} \otimes \mathcal{L}'$ is relatively ample over Y.

   Solution: For any point $x \in X$ take open affine neighborhoods U, $U' \subset X$ such that $\mathcal{L}|_{f^{-1}(U)}$ and $\mathcal{L}'|_{f^{-1}(U')}$ are ample. Choose an open affine neighborhood $V \subset U \cap U'$. Once we know that $\mathcal{L}|_{f^{-1}(V)}$ and $\mathcal{L}'|_{f^{-1}(V)}$ are ample, we deduce $\mathcal{L}|_{f^{-1}(V)} \otimes \mathcal{L}'|_{f^{-1}(V)} \cong (\mathcal{L} \otimes \mathcal{L}')|_{f^{-1}(V)}$ is ample by Sheet 4, Exercise 3c. (Parts (b) and (c) of that exercise do not require the noetherian hypothesis. Quasicompact and (quasi)separated suffice.) Hence $\mathcal{L} \otimes \mathcal{L}'$ is relatively ample over Y.

   By symmetry it remains to show that $\mathcal{L}|_{f^{-1}(V)}$ is ample. For this note that $f^{-1}(U)$ is quasicompact by assumption. Also, since it possesses an ample invertible sheaf, by [Görtz and Wedhorn, Remark 13.61(3)] it embeds into $\text{Proj} \bigoplus_{n \geqslant 0} f_* \mathcal{L}^{\otimes n}(U)$; hence it is separated. We can then apply the following lemma to $Z := f^{-1}(U)$:

   Lemma (Liu, Lemma 5.1.35b). *Let Z be separated and quasicompact and let $\mathcal{L}$ be an ample invertible sheaf on Z. If $V \subset Z$ is an open quasicompact subscheme, then $\mathcal{L}|_V$ is ample.*

3. Let $\mathcal{S} = \bigoplus_{d \geqslant 0} \mathcal{S}_d$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Then for each open affine $U \subset X$, we have the graded $\mathcal{O}_X(U)$-modules $\mathcal{S}(U) = \bigoplus_{d \geqslant 0} \mathcal{S}_d(U)$.

   (a) Show that there exists a natural scheme $\pi\colon \text{Proj}\, \mathcal{S} \to X$ together with isomorphisms $\eta_U\colon \pi^{-1}(U) \xrightarrow{\sim} \text{Proj}\, \mathcal{S}(U)$ over U for all open affine subschemes $U \subset X$.

   (b) Suppose $\mathcal{S}$ is locally generated over $\mathcal{S}_0$ by $\mathcal{S}_1$. Show that there is a natural invertible sheaf $\mathcal{O}(1)$ on $\text{Proj}\, \mathcal{S}$ whose restriction to each $\pi^{-1}(U)$ corresponds to the previously known sheaf $\mathcal{O}(1)$ on $\text{Proj}\, \mathcal{S}(U)$ via η_U. We call $\text{Proj}\, \mathcal{S}$ the *relative* Proj of $\mathcal{S}$.

(c) Show that for any projective morphism $f\colon Y \to X$ with relatively very ample invertible sheaf $\mathcal{L}$ on Y there is a natural isomorphism $Y \cong \operatorname{Proj} \bigoplus_{d\geqslant 0} f_*\mathcal{L}^{\otimes d}$.

Solution:

(a) and (b) See https://stacks.math.columbia.edu/tag/01NM. The main thing to prove is the following: Let S be a $\mathbb{Z}^{\geqslant 0}$-graded ring and let $A := S_0$. Let B be an A-algebra. Let $S_B := S \otimes_A B$. Then $\operatorname{Proj}(S_B) \cong \operatorname{Proj} S \times_{\operatorname{Spec} A} \operatorname{Spec} B$. Such an isomorphism is unique by the universal property of the fiber product. Suppose S is generated by S_1 as an A-algebra. Then $\mathcal{O}_{\operatorname{Proj} S_B}(1)$ is isomorphic to the pullback of $\mathcal{O}_{\operatorname{Proj} S}(1)$ under the canonical morphism $\operatorname{Proj} S_B \to \operatorname{Proj} S$. These statements allow us to construct $\operatorname{Proj} \mathcal{S}$ and $\mathcal{O}(1)$ by glueing.

(c) Let $U = \operatorname{Spec} A \subset X$ be an open subscheme such that $\mathcal{L}|_{f^{-1}(U)}$ is very ample. Thus $\mathcal{L}|_{f^{-1}(U)}$ corresponds to $\mathcal{O}(1)$ for an isomorphism $f^{-1}(U) \cong \operatorname{Proj} R$ over U. By Sheet 4, Exercise 1, we obtain a commutative diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\sim} & \operatorname{Proj} \bigoplus_{d\geqslant 0} \mathcal{L}^{\otimes d}(f^{-1}(U)) \\ \cong\downarrow & & \cong\downarrow \\ \operatorname{Proj} R & \xrightarrow{\sim} & \operatorname{Proj} R', \end{array}$$

and thus a natural isomorphism over U

$$f^{-1}(U) \;\cong\; \operatorname{Proj} \bigoplus_{d\geqslant 0} \mathcal{L}^{\otimes d}(f^{-1}(U)) \;=\; \operatorname{Proj}\Big(\bigoplus_{d\geqslant 0} f_*\mathcal{L}^{\otimes d}\Big)(U).$$

We vary U and glue these isomorphisms to construct the desired isomorphism.

4. For any integer $n \geqslant 0$ consider the morphism

$$X := \mathbb{A}^{n+1} \smallsetminus \{0\} \xrightarrow{\pi} Y := \mathbb{P}^n, \quad (x_0, \ldots, x_n) \mapsto (x_0 : \ldots : x_n).$$

Prove that π is affine and determine the sheaf of $\mathcal{O}_X$-algebras $\pi_*\mathcal{O}_X$.

Solution: We have a covering of $\mathbb{P}^n$ by the standard open affines $\overline{D}_{x_i}$. Since $\pi^{-1}(\overline{D}_{x_i})$ is isomorphic to the standard affine open $D_{x_i} \subset \mathbb{A}^{n+1}$, if follows from Exercise 5b that π is affine.

We now determine $\pi_*\mathcal{O}_X$. By Sheet 3, Exercise 3, it is quasicoherent. The restriction $\pi_*\mathcal{O}_X|_{\overline{D}_{xi}}$ is thus determined by its global sections. Let $S := k[x_0, \ldots, x_n]$. We have

$$\pi_*\mathcal{O}_X(\overline{D}_{x_i}) = \mathcal{O}_X(\pi^{-1}(\overline{D}_{x_i})) = S_{x_i} = \bigoplus_{d\in\mathbb{Z}} S_{x_i,0} \cdot x_i^d = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)(\overline{D}_{x_i}).$$

By gluing it follows that $\pi_*\mathcal{O}_X = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)$.

5. (a) Prove that a scheme X is affine if and only if there exist sections $s_1, \ldots, s_n \in \mathcal{O}_X(X)$ generating the unit ideal such that for each i the open subset $D_{s_i} = X \smallsetminus V(s_i)$ is affine.

   (b) Prove that for any morphism $f : X \to Y$, the following are equivalent:

   i. There exists an affine open covering $Y = \bigcup V_i$ such that each $f^{-1}(V_i)$ is affine.

   ii. For every open affine $V \subset Y$ the inverse image $f^{-1}(V)$ is affine.

Solution:

Proof of (a): "$\Rightarrow$": Take $s = 1 \in \mathcal{O}_X(X)$. "$\Leftarrow$": Let $A := \mathcal{O}_X(X)$. Let $\varphi\colon X \to \operatorname{Spec} A$ be the natural structure morphism. For each i, this restricts to the morphism $\varphi_i\colon D_{s_i} \to \operatorname{Spec} A_{s_i}$ of affine schemes determined by the homomorphism

$$\varphi_i^\flat\colon A_{s_i} \to \mathcal{O}_X(D_{s_i}),\ \frac{f}{s_i^n} \mapsto \frac{f|_{D_{s_i}}}{s_i^n|_{D_{s_i}}}.$$

Just as in the solution to Sheet 4, Exercise 1, we use parts (a) and (b) of the lemma in §5.7 to show that $\varphi_i^\flat$, and hence φ_i, is an isomorphism. Since the D_{s_i} and $\operatorname{Spec} A_{s_i}$ cover X and $\operatorname{Spec} A$ respectively, it follows that φ is an isomorphism and X is affine.

We note that the lemma requires that X be quasicompact and separated. But X is a finite union of quasicompact open subschemes and hence quasicompact. The fact that X is separated follows from the fact that separatedness is local on the target and that the φ_i are morphisms of affine schemes, hence separated.

Proof of (b): (ii)$\Rightarrow$ (i) is clear. For (i) $\Rightarrow$ (ii), cover each $V \cap V_i$ by open affines V_{ij} which are standard open in both V and V_i. Then each $f^{-1}(V_{ij})$ is affine. If V_{ij} is the standard open of V associated to $s_{ij} \in \mathcal{O}_V(V)$, then $f^{-1}(V_{ij})$ is the standard open of $f^{-1}(V)$ associated to f^*s_{ij}, in the sense of §5.7. Now apply (a).

*6. (*Vector bundles versus locally free sheaves*)

A *vector bundle of rank $n \in \mathbb{Z}^{\geqslant 0}$ over X* is a morphism $f\colon V \to X$ together with morphisms $+\colon V \times_X V \to V$ and $\cdot\colon \mathbb{A}^1 \times V \to V$ and $0\colon X \to V$ over X, such that there exists an open covering $X = \bigcup_\alpha U_\alpha$ and isomorphisms $f^{-1}(U_\alpha) \cong \mathbb{A}^n \times U_\alpha$ over U_α, such that the morphisms $+, \cdot, 0$ correspond to the morphisms

$$\begin{aligned}
\mathbb{A}^n \times \mathbb{A}^n \times U_\alpha &\to \mathbb{A}^n \times U_\alpha, & ((x_1, \ldots, x_n), (y_1, \ldots, y_n), u) &\mapsto ((x_1 + y_1, \ldots, x_n + y_n), u) \\
\mathbb{A}^1 \times \mathbb{A}^n \times U_\alpha &\to \mathbb{A}^n \times U_\alpha, & (\lambda, (x_1, \ldots, x_n), u) &\mapsto ((\lambda x_1, \ldots, \lambda x_n), u) \\
U_\alpha &\to \mathbb{A}^n \times U_\alpha, & u &\mapsto ((0, \ldots, 0), u)
\end{aligned}$$

Homomorphisms of vector bundles are morphisms of schemes over X which make certain commutative diagrams with the morphisms $+, \cdot, 0$ that one can guess. A vector bundle of rank 1 is called a *line bundle.*

(a) Look up the definition of vector bundles on a differentiable manifold and compare.

(b) Write down the commutative diagrams for morphisms of vector bundles.

(c) Why do the above conditions not include analogues of the usual vector space axioms? What would these analogues say?

(d) For any vector bundle V define the sheaf of sections $\mathcal{V}$ as a sheaf of $\mathcal{O}_X$-modules and extend this to a functor.

(e) Prove that this induces an equivalence from the category of vector bundles of all ranks to the full subcategory of quasi-coherent sheaves on X that are locally free of some rank.

(f) Discuss what is wrong with [Görtz and Wedhorn, Definition 11.5]. Construct a concrete example to show that (e) does not hold with that definition.

(g) Promise to never confuse vector bundles with locally free sheaves, or line bundles with invertible sheaves, even if many other people do.

Solution:

(a) See for example [Lee, John M. *Introduction to Smooth Manifolds*, Springer 2003.]

(b) A morphism of vector bundles $V \to V'$ over X is a morphism $\varphi\colon V \to V'$ of schemes over X which is compatible with $+, \cdot, 0$, namely, such that the following three diagrams commute:

$$\begin{array}{ccc} V \times_X V & \xrightarrow{+} & V \\ \downarrow{\scriptstyle \varphi\times\varphi} & & \downarrow{\scriptstyle \varphi} \\ V' \times_X V' & \xrightarrow{+} & V' \end{array} \qquad \begin{array}{ccc} \mathbb{A}^1 \times V & \xrightarrow{\cdot} & V \\ \downarrow{\scriptstyle \mathrm{id}\times\varphi} & & \downarrow{\scriptstyle \varphi} \\ \mathbb{A}^1 \times V' & \xrightarrow{\cdot} & V' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{0} & V \\ & \searrow{\scriptstyle 0} & \downarrow{\scriptstyle \varphi} \\ & & V' \end{array}$$

(c) As a typical example we discuss the axiom for associativity. On the standard affine space $\mathbb{A}^n$ it states that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{A}^n \times \mathbb{A}^n \times \mathbb{A}^n & \xrightarrow{+\times\mathrm{id}} & \mathbb{A}^n \times \mathbb{A}^n \\ {\scriptstyle \mathrm{id}\times+}\downarrow & & {\scriptstyle +}\downarrow \\ \mathbb{A}^n \times \mathbb{A}^n & \xrightarrow{+} & \mathbb{A}^n. \end{array}$$

For the vector bundle $V := \mathbb{A}^n \times X$ this translates into the commutative diagram

$$\begin{array}{ccc} V \times_X V \times_X V & \xrightarrow{+\times\mathrm{id}} & V \times_X V \\ {\scriptstyle \mathrm{id}\times+}\downarrow & & {\scriptstyle +}\downarrow \\ V \times_X V & \xrightarrow{+} & V. \end{array}$$

But this diagram is well-defined for an arbitrary vector bundle, and it commutes because it commutes after pullback to an open covering of X.

(d) For each open $U \subset X$, we define $\mathcal{V}(U) := \mathrm{Hom}_X(U, V)$, the set of morphisms $s\colon U \to V$ such that $f \circ s$ is the inclusion $U \hookrightarrow X$. With the restriction of morphisms this constitutes a sheaf of sets on X. The morphisms $+, \cdot, 0$ turn $\mathcal{V}(U)$ into an $\mathcal{O}_X(U)$-module whenever $U \subset U_\alpha$ for some α. Being a sheaf, it becomes a sheaf of $\mathcal{O}_X$-modules.

Any morphism of vector bundles $\varphi\colon V' \to V$ over X determines maps $\mathcal{V}'(U) \to \mathcal{V}(U)$, $s \mapsto \varphi \circ s$ and hence a morphism of sheaves $\mathcal{V}' \to \mathcal{V}$. The commutative diagrams in (b) imply that this is a homomorphism of $\mathcal{O}_X$-modules. It clearly sends the identity to the identity and is compatible with composition; so it defines a functor from the category of vector bundles over X to the category of $\mathcal{O}_X$-modules.

(e) The isomorphisms $f^{-1}(U_\alpha) \cong \mathbb{A}^n \times U_\alpha$ show that $\mathcal{V}$ is locally free of finite rank since

$$\mathrm{Hom}_U(U_\alpha, \mathbb{A}^n_U) = \mathcal{O}_U(U)^n.$$

Hence the functor $V \mapsto \mathcal{V}$ lands in the correct category.

We define a functor in the opposite direction as follows. First one defines the notion of the *relative* Spec of a quasicoherent sheaf of $\mathcal{O}_X$-algebras in analogy with the relative Proj from Exercise 3. One shows that if $\mathcal{E}$ is a locally free sheaf of finite rank on X, then $E := \mathrm{Spec}\big(\mathrm{Sym}(\mathcal{E}^\vee)\big)$ is a vector bundle and that $\mathcal{E} \mapsto E$ is a functor. This is all discussed in Görtz and Wedhorn §11.4, which still applies despite the flaw in their definition.

Aliter: Take a locally free sheaf $\mathcal{E}$ of rank n on X. Choose an open covering $X = \bigcup_\alpha U_\alpha$ and isomorphisms $\mathcal{E}|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha}^{\oplus n}$ for all α. Then for any α, β the two isomorphisms over $U_\alpha \cap U_\beta$ differ by an element of $\mathrm{GL}_n(\mathcal{O}_X(U_\alpha))$. Use these to glue the "constant" vector bundles $\mathbb{A}^n \times U_\alpha$ to a vector bundle V over X.

(f) The problem is that the isomorphisms $c_i\colon V|_{U_i} \to \mathbb{A}^n_{Ui}$ are not fixed as part of the data of a vector bundle. We can therefore modify them by arbitrary scheme automorphisms of $\mathbb{A}^n_{U_i}$ over U_i. For instance by translation with a section $U_i \to \mathbb{A}^n_{Ui}$; thus V does not come with a given zero section; and this alone prevents one to turn its sheaf of sections functorially into a sheaf of abelian groups. Also $\mathbb{A}^n_{U_i}$ can have automorphisms over U_i which are not linear, such as $(x, y) \mapsto (x, y + x^{42})$. Thus even if one were given the zero section, one would still not have a natural addition law on V. Furthermore, even if one were given the zero section and the addition law, this might not determine the scalar multiplication, because in characteristic $p > 0$ the automorphism $(x, y) \mapsto (x, y + x^p)$ is a group isomorphism but does not commute with scalar multiplication. (Compare the definition of differentiable manifolds, where a differentiable atlas is part of the structure.)

(g) I solemnly swear to never confuse vector bundles with locally free sheaves.