Solutions 6

Picard Group, Divisors, Divisor Classes

Let k be a field.

1. Prove that $\mathrm{Aut}_k(\mathbb{P}^n_k)$ is naturally isomorphic to $\mathrm{PGL}_{n+1}(k)$.

   Solution: See Hartshorne, Example II.7.1.1.

2. Consider a morphism $\varphi\colon \mathbb{P}^n_k \to \mathbb{P}^m_k$ over k which does not factor through a k-valued point of $\mathbb{P}^m_k$. Show that $n \leqslant m$ and that φ is the composite of the d-uple embedding $\mathbb{P}^n_k \to \mathbb{P}^N_k$ for a uniquely determined $d \geqslant 1$ and a morphism $\mathbb{P}^N_k - L \to \mathbb{P}^m_k$ induced by linear polynomials $\ell_0, \ldots, \ell_m$, where L is a linear subspace of $\mathbb{P}^N_k$. Deduce that φ has finite fibers and that $\dim \varphi(\mathbb{P}^n_k) = n$.

   Solution: Giving a morphism $\mathbb{P}^n_k \to \mathbb{P}^m_k$ over k corresponds to giving an invertible sheaf $\mathcal{L}$ on $\mathbb{P}^n_k$ along with global sections $f_0, \ldots, f_m$ generating it. Since $\mathrm{Pic}(\mathbb{P}^n_k)$ is generated by the class of $\mathcal{O}(1)$, we can without loss of generality assume that $\mathcal{L} = \mathcal{O}(d)$ for some integer d. This d is unique except if $n = 0$, but in that case $\mathbb{P}^n_k \cong \operatorname{Spec} k$, so the morphism factors through a k-valued point, which was ruled out by assumption. Then $f_0, \ldots, f_m$ are homogenous polynomials of degree d which are not all 0; hence $d \geqslant 0$. If $d = 0$, the f_i are constant and φ factors through the closed point $(f_0 : \ldots : f_m) \in \mathbb{P}^m(k)$, which was ruled out by assumption. Thus $d \geqslant 1$. Since the f_i generate the invertible sheaf $\mathcal{O}(d)$, their joint zero locus $Y := V(f_0, \ldots, f_m)$ must be empty. As $\dim Y \geqslant n-(m+1)$, it follows that $n \leqslant m$.

   Now recall that the d-uple embedding $\nu_d\colon \mathbb{P}^n_k \to \mathbb{P}^N_k$ is given by the same invertible sheaf $\mathcal{O}(d)$ on $\mathbb{P}^n_k$ but with all monomials of degree d as sections. Since each f_i is a linear combination of such monomials, there are linear forms $\ell_0, \ldots, \ell_m$ on $\mathbb{P}^N_k$ such that $\nu_d^* \ell_i = f_i$ for all i. Let $L := V(\ell_0, \ldots, \ell_m) \subset \mathbb{P}^N_k$. The ℓ_i generate $\mathcal{O}_{\mathbb{P}^N_k}(1)$ over $\mathbb{P}^N_k \setminus L$ and define a linear projection $p\colon \mathbb{P}^N_k \setminus L \to \mathbb{P}^m_k$ such that $p^* X_i = \ell_i$ for each $i = 0, \ldots, m$. Since $\nu_d^{-1}(L) = Y = \varnothing$, it follows that ν_d factors through $\mathbb{P}^N_k \setminus L$. We have $(p \circ \nu_d)^* X_i = f_i$ for all i, and thus $\varphi = p \circ \nu_d$.

   To show that φ has finite fibers, it is sufficient to show that $p \circ \iota$, where $\iota\colon \nu_d(\mathbb{P}^n_k) \hookrightarrow \mathbb{P}^N_k$ is the canonical embedding, has finite fibers. Let $x \in \mathbb{P}^m_k$. Then $x \in D^{\mathbb{P}^m_k}_{X_i}$ for some i and $p^{-1}(D^{\mathbb{P}^m_k}_{X_i}) = D^{\mathbb{P}^N_k}_{\ell_i}$, which is affine. It follows that

$$(p \circ \iota)^{-1}(x) = p^{-1}(x) \cap \nu_d(\mathbb{P}^n_k) \subset D^{\mathbb{P}^N_k}_{\ell_i}$$

   is affine. But it is also closed in the projective variety $\nu_d(\mathbb{P}^n_k)$ and hence projective. It follows that $(p \circ \iota)^{-1}(x)$ is finite, as desired. Finally, we conclude that

$\dim \varphi(\mathbb{P}^n_k) = n$ by noting that $\dim \mathbb{P}^n_k - \dim \varphi(\mathbb{P}^n_k)$ is bounded above by the dimension of any non-empty fiber of φ [Hartshorne, Exercise II.3.22], which we have just shown is equal to 0.

3. Consider the nodal cubic curve $X := V(C(C-B)A-B^3) \subset \mathbb{P}^2_k$. Prove that $\mathrm{Pic}(X) \cong k^\times \times \mathbb{Z}$.
(Hint: Recall that X has the normalization $\pi\colon \mathbb{P}^1_k \twoheadrightarrow X$. To describe an invertible sheaf $\mathcal{L}$ on X, describe $\pi^*\mathcal{L}$ and find out which additional information is necessary to determine $\mathcal{L}$.)

Solution: In §5.8 we constructed $\pi\colon \tilde{X} := \mathbb{P}^1_k \twoheadrightarrow X$ such that the inverse image of the singular point P_0 with coordinates $(A:B:C) = (1:0:0)$ is the reduced closed subscheme $\{0,1\} \subset \tilde{X}$ and that π induces an isomorphism $\tilde{X} \smallsetminus \{0,1\} \xrightarrow{\sim} X \smallsetminus \{P_0\}$. We also constructed affine charts which show that

$$\mathcal{O}_X \;\cong\; \{f \in \pi_*\mathcal{O}_{\tilde{X}} \mid f(1) = f(0)\}. \tag{3.1}$$

Now observe that for any invertible sheaf $\mathcal{L}$ on X, the pullback $\pi^*\mathcal{L}$ is an invertible sheaf on $\tilde{X}$ and π induces isomorphisms of 1-dimensional k-vector spaces

$$(\pi^*\mathcal{L})_0 \otimes_{\mathcal{O}_{\tilde{X},0}} k \;\overset{\sim}{\longleftarrow}\; \mathcal{L}_{P_0} \otimes_{\mathcal{O}_{X,P_0}} k \;\overset{\sim}{\longrightarrow}\; (\pi^*\mathcal{L})_1 \otimes_{\mathcal{O}_{\tilde{X},0}} k. \tag{3.2}$$

Since $\mathrm{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$ with generator $\mathcal{O}(1) \cong \mathcal{O}_{\tilde{X}}(\infty)$, we can choose an isomorphism $u\colon \pi^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_{\tilde{X}}(d\cdot\infty)$ for a unique integer d. Then the composite isomorphism in (3.2) amounts to an isomorphism

$$k \;=\; \mathcal{O}_{\tilde{X},0} \otimes_{\mathcal{O}_{\tilde{X},0}} k \;=\; \mathcal{O}_{\tilde{X}}(d\cdot\infty)_0 \otimes_{\mathcal{O}_{\tilde{X},0}} k \;\overset{\sim}{\longrightarrow}\; \mathcal{O}_{\tilde{X}}(d\cdot\infty)_1 \otimes_{\mathcal{O}_{\tilde{X},0}} k \;=\; k. \tag{3.3}$$

This is multiplication by an element $\lambda \in k^\times$. Note that u is unique up to an automorphism of $\mathcal{O}_{\tilde{X}}(d\cdot\infty)$, and any automorphism of an invertible sheaf is multiplication by a nowhere vanishing section of the structure sheaf. In this case u is therefore unique up to a scalar $\Gamma(\tilde{X}, \mathcal{O}^\times_{\tilde{X}}) = k^\times$. Since this scalar appears equally on both sides of (3.2), the scalar λ is independent of the choice of u. A similar calculation shows that λ depends only on the isomorphism class of $\mathcal{L}$. Together this therefore defines a map $\mathrm{Pic}(X) \to k^\times \times \mathbb{Z}$, $[\mathcal{L}] \mapsto (\lambda, d)$. As the construction is compatible with tensor product, this map is a homomorphism.

If $[\mathcal{L}]$ lies in the kernel, we have $d = 0$ and $\lambda = 1$. Then $\pi^*\mathcal{L} \cong \mathcal{O}_{\tilde{X}}$ and (3.1) shows that $\mathcal{L} \cong \mathcal{O}_X$. Thus the homomorphism is injective. For arbitrary $(\lambda, d) \in k^\times \times \mathbb{Z}$ set

$$\mathcal{L} \;:=\; \{f \in \pi_*\mathcal{O}_{\tilde{X}}(d\cdot\infty) \mid f(1) = \lambda\cdot f(0)\}. \tag{3.4}$$

Using (3.1) and a quick local calculation one finds that this is an invertible sheaf on X which gives back the pair (λ, d). Thus the homomorphism is surjective, and hence an isomorphism, as desired.

Aliter: Follow Hartshorne Example II.6.11.4 and Exercises II.6.7 and II.6.9.

*4. Determine the Picard group of the cuspidal cubic curve $V(Y^2Z - X^3) \subset \mathbb{P}^2_k$.

Solution: Here again the normalization of $X := V(Y^2Z - X^3)$ is isomorphic to $\mathbb{P}^1_k$, but the inverse image of the unique singular point $P_0 = (0:0:1)$ is a k-valued point with multiplicity 2. Follow Hartshorne Exercise II.6.9.

5. (a) For any noetherian integral scheme X and any irreducible closed subscheme $Y \subset X$ of codimension 1, construct a natural exact sequence $\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X \smallsetminus Y) \to 0$.

   (b) Prove that $\mathrm{Cl}(\mathbb{P}^n_k \smallsetminus V(f)) \cong \mathbb{Z}/d\mathbb{Z}$ for any irreducible homogeneous polynomial f of degree $d > 0$.

Solution: See Hartshorne, Proposition II.6.5 and Example II.6.5.1.

6. For any locally factorial noetherian separated integral scheme X and any $n \geqslant 0$ prove that there are natural isomorphisms

   (a) $\mathrm{Cl}(X \times \mathbb{A}^n) \cong \mathrm{Cl}(X)$ and

   (b) $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z}$.

Solution: For (a) see Hartshorne, Proposition II.6.6 when $n = 1$; the general case follows by induction. For (b) let $H := X \times V(X_n)$ denote the hyperplane at infinity in $X \times \mathbb{P}^n$. Then $(X \times \mathbb{P}^n) \smallsetminus H \cong X \times \mathbb{A}^n$, and by Exercise 5a, we obtain an exact sequence

$$\mathbb{Z} \xrightarrow{i} \mathrm{Cl}(X \times \mathbb{P}^n) \xrightarrow{j} \mathrm{Cl}(X \times \mathbb{A}^n) \longrightarrow 0.$$

Let η_X be the generic point of X. The morphism $\varphi\colon \mathbb{P}^n_{K(X)} \cong \eta_X \times \mathbb{P}^n \hookrightarrow X \times \mathbb{P}^n$ is dominant and, as we showed in §5.9, thus induces a homomorphism $\varphi^*\colon \mathrm{DivCl}(X \times \mathbb{P}^n) \to \mathrm{DivCl}(\mathbb{P}^n_{K(X)}) \cong \mathbb{Z}$. Since $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{DivCl}(X \times \mathbb{P}^n)$, we obtain a morphism $r\colon \mathrm{Cl}(X \times \mathbb{P}^n) \to \mathbb{Z}$. Moreover, since H corresponds to a hyperplane in $\mathbb{P}^n_{K(X)}$, it follows that $r([H]) = 1$. Thus $r \circ i = \mathrm{id}_{\mathbb{Z}}$, so i is injective and the sequence is split. Using part (a), we obtain $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z}$.

7. Which of the following divisors is principal, resp. ample, resp. very ample, resp. equivalent to an effective divisor?

   (a) $D = V(X^3 + Y^3 + Z^3) - V(X^2 + Y^2 + Z^2)$ on $\mathbb{P}^2_k$.

   (b) $D = -P$ for $P := V(x-1, y)$ on $X := \operatorname{Spec} \mathbb{R}[x,y]/(x^2 + y^2 - 1)$.

   (c) $D = a \operatorname{diag} \mathbb{P}^1_k + b \operatorname{pr}_1^* P + c \operatorname{pr}_2^* P$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ for $P \in \mathbb{P}^1(k)$ and $a, b, c \in \mathbb{Z}$.

Solution:

(a) For any non-zero homogeneous polynomial $f \in k[X,Y,Z]$ of degree $d > 0$ the quotient $\frac{f}{Z^d}$ is a non-zero rational function on $\mathbb{P}^2_k$ with divisor $\mathrm{div}(\frac{f}{Z^d}) = V(f) -$

$dV(Z)$. Thus D is equivalent to the effective divisor $3V(Z) - 2V(Z) = V(Z)$. As that is ample and very ample, so is D. Since $[V(Z)]$ is the generator of $\mathrm{Cl}(\mathbb{P}^2_k) \cong \mathbb{Z}$, it follows that D is not principal.

(b) Here $X \cong \mathbb{P}^1_{\mathbb{R}} \setminus V(U^2 + V^2)$ where $I := V(U^2 + V^2)$ is a closed point with residue field $\mathbb{R}(I) \cong \mathbb{C}$ of degree 2; hence $\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$ by Exercise 5b above. Also P is a closed point with residue field $\mathbb{R}$, hence of degree 1; so it represents the non-trivial element of $\mathrm{Cl}(X)$ of order 2. Thus $D = -P$ is equivalent to the effective divisor P, but is not principal. On the other hand any invertible sheaf on an affine scheme is ample; hence D is ample. Since X is of finite type over k it follows that every sufficiently large tensor power of $\mathcal{O}(D)$ is very ample. Thus for odd $n \in \mathbb{Z}^{>0}$ large enough, the divisor nD is very ample. Since D has order 2, it follows that $nD \sim D$; hence D is very ample.

(c) The solution to Exercise 6b for $X = \mathbb{P}^1_k$ and the fact that $\mathrm{Cl}(\mathbb{P}^1_k) \cong \mathbb{Z}$ together show that $\mathrm{Cl}(\mathbb{P}^1_k \times \mathbb{P}^1_k) \cong \mathbb{Z}^2$ with generators $[\mathrm{pr}_1^* P]$ and $[\mathrm{pr}_2^* P]$. Identify the function fields of the two copies of $\mathbb{P}^1_k$ with $k(x)$ and $k(y)$, respectively, where P corresponds to the point $x = y = \infty$. Then a direct calculation on open charts shows that $\mathrm{div}(x - y) = \mathrm{diag}\, \mathbb{P}^1_k - \mathrm{pr}_1^* P - \mathrm{pr}_2^* P$. Therefore $[D] = (a + b)[\mathrm{pr}_1^* P] + (a + c)[\mathrm{pr}_2^* P]$. Thus D is principal if and only if $a + b = a + c = 0$, and it is equivalent to an effective divisor if $a + b,\ a + c \geqslant 0$.

For the converse and the other properties we use the fact that $\mathcal{O}(P) \cong \mathcal{O}(1)$ on $\mathbb{P}^1_k$. Thus $\mathcal{O}(D) \cong \mathrm{pr}_1^* \mathcal{O}(a + b) \otimes \mathrm{pr}_2^* \mathcal{O}(a + c)$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$. A quick calculation shows that

$$(*) \qquad \Gamma(\mathbb{P}^1_k \times \mathbb{P}^1_k, \mathcal{O}(D)) \cong \Gamma(\mathbb{P}^1_k, \mathcal{O}(a + b)) \otimes_k \Gamma(\mathbb{P}^1_k, \mathcal{O}(a + c)).$$

If D is equivalent to an effective divisor D', then $\mathcal{O}(D) \cong \mathcal{O}(D')$ possesses a non-zero global section; hence both $\mathcal{O}(a + b)$ and $\mathcal{O}(a + c)$ must possess a non-zero global section on $\mathbb{P}^1_k$. We already know that this requires $a + b,\ a + c \geqslant 0$. Thus D is equivalent to an effective divisor if and only if $a + b,\ a + c \geqslant 0$.

If $a + b,\ a + c > 0$, then $\mathcal{O}(a + b)$ and $\mathcal{O}(a + c)$ are very ample on $\mathbb{P}^1_k$; hence by the Segre embedding D is very ample. If by contrast $a + b \leqslant 0$, every section in $(*)$ is constant on the first factor $\mathbb{P}^1_k$; hence D cannot be very ample. The same argument works in the second factor; thus D is very ample if and only if $a + b$, $a + c > 0$.

Finally, since $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is separated of finite type over k, the divisor D is ample if and only if some positive multiple nD is very ample. We have just seen that this is equivalent to $n(a + b),\ n(a + c) > 0$; hence to $a + b,\ a + c > 0$; so again D is ample if and only if $a + b,\ a + c > 0$.