Solutions 7

Divisors, Divisor Classes, Differentials

1. Consider the nonsingular cubic curve $X := V(Y^2Z - X^3 - 4XZ^2) \subset \mathbb{P}^2_{\mathbb{C}}$ and its closed points $P_0 := (0:1:0)$ and $P := (2:4:1)$ and $Q := (0:0:1)$. Which of the following divisors on X is principal, resp. equivalent to an effective divisor?

   (a) $D_1 := P - P_0$.

   (b) $D_2 := P + Q - P_0$.

   (c) $D_3 := 2P + Q - 3P_0$.

   Solution:

   (a) Since $P \neq P_0$, by a theorem in the lecture D_1 represents a non-trivial element of $\mathrm{Cl}^0(X)$, hence D_1 is not principal. Since $\deg(D_1) = 0$, any equivalent effective divisor D' also has degree 0, so it must be empty, hence D_1 must be principal, contradiction. Thus D_1 is not equivalent to an effective divisor.

   (b) Since $\deg(D_2) = 1$ the divisor D_2 is not principal. By a theorem in the lecture there exists $R \in X$ with $[P - P_0] + [Q - P_0] = [R - P_0]$. Thus $[P + Q - P_0] = [R]$, so D_2 is equivalent to the effective divisor R.

   (c) Since $\deg(D_3) = 0$, the same argument as in (a) shows that D_3 is equivalent to an effective divisor if and only if it is principal. To decide that, take the line $L = V(Y - 2X)$ through P and Q and compute the point R with $L \cap X = P + Q + R$, which by substituting $Y = 2X$ in the equation $Y^2Z - X^3 - 4XZ^2$ and factoring $(2X)^2Z - X^3 - 4XZ^2 = -X(X - 2Z)^2$ comes out as $R = P$ again. Thus $[2P + Q] = [3P_0]$; hence D_3 is principal.

2. Let X be as in the previous exercise. Recall that its set of closed points $|X|$ possesses a natural abelian group structure with zero element P_0.

   (a) Show that $P \in |X| \setminus \{P_0\}$ has order 2 in $|X|$ if and only if the tangent line at P passes through P_0.

   (b) An *inflection point* of a plane curve is a nonsingular point P of the curve, whose tangent line has intersection multiplicity $\geqslant 3$ with the curve at P. Show that the inflection points in $|X|$ are precisely P_0 and all points of order 3.

   Solution: For clarity let us denote the group law on $|X|$ by $\oplus$. We claim that three points $P, Q, R \in |X|$ satisfy $P \oplus Q \oplus R = P_0$ if and only if there exists a line $L \subset \mathbb{P}^2_k$ such that $L \cap X = P + Q + R$ as a divisor with multiplicities. We have seen the "if" part of this in the lecture. For the "only if" part suppose that $P \oplus Q \oplus R = P_0$. Let $L \subset \mathbb{P}^2_k$ be the unique line connecting P and Q if these are distinct, respectively the tangent at X through $P = Q$ otherwise. Then $L \cap X = P + Q + R'$ for a point $R \in |X|$. By the "if" part we then have $P \oplus Q \oplus R' = P_0 = P \oplus Q \oplus R$ and by the uniqueness of inverses in the group therefore $R' = R$, proving the "only if" part.

(a) A point $P \neq P_0$ has order 2 if and only if $P \oplus P \oplus P_0 = P_0$. By the preliminary claim, this means that there is a line $L \subset \mathbb{P}^2_k$ with $L \cap X = 2P + P_0$, which is equivalent to saying that the tangent line at P passes through P_0.

(b) A point $P \in |X|$ is an inflection point if and only if there is a line $L \subset \mathbb{P}^2_k$ with $L \cap X = 3P$. By the preliminary claim this is equivalent to $P \oplus P \oplus P = P_0$. But this means that $P = P_0$ or that P has order 3.

**3. Let $X = \operatorname{Spec} k[S,T,U]/(UT - S^2)$, and let C be the closed subscheme $V(S,T)$ of X.

(a) Show that $C \cong \mathbb{A}^1_k$ is a prime Weil divisor on X and that $\mathrm{Cl}(X)$ is a group of order 2 which is generated by the class of C.

(b) Show that $\mathrm{DivCl}(X) = 0$ and deduce that $\mathrm{cyc}\colon \mathrm{Div}(X) \to Z^1(X)$ is not surjective.

4. Consider ring homomorphisms $A \to B \to C$. Prove the following statements from the course:

(a) For any multiplicative system $S \subset B$, there is a natural isomorphism of $S^{-1}B$-modules $S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A}$.

(b) There is a natural exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

(c) If $C = B/J$ for an ideal $J \subset B$, there is a natural exact sequence

$$J/J^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0.$$

Solution: See Liu, Proposition 6.1.8.

5. Compute $\Omega_{B/A}$ in the following cases, and determine the largest open subscheme of $\operatorname{Spec} B$ over which $\Omega_{B/A}$ is locally free.

(a) $B = A[X,Y]/(XY - t)$ for a discrete valuation ring A with uniformizer t.

(b) $B = k[X,Y,Z,W]/(XY - ZW)$ where $A = k$ is a field.

(c) $B = k[X]/(X^n)$ for a field $A = k$ and an integer $n \geqslant 1$.

Solution: (a) Let $J := (XY - t) \subset B' := A[X, Y]$; then the second exact sequence $J/J^2 \to B \otimes_{B'} \Omega_{B'/A} \to \Omega_{B/A} \to 0$ yields an exact sequence of B-modules

$$B \longrightarrow B \cdot dX \oplus B \cdot dY \longrightarrow \Omega_{B/A} \longrightarrow 0$$

where the homomorphism on the left sends 1 to $d(XY - t) = Y \cdot dX + X \cdot dY$. After inverting X the image of that homomorphism becomes the direct summand generated by $\frac{Y}{X} \cdot dX + dY$; hence the quotient becomes free of rank 1. The same arguments applies when inverting Y. But modulo the maximal ideal $\mathfrak{m} := (X, Y) = (X, Y, t)$ the homomorphism becomes zero; so the cokernel is not locally free of rank 1 there. The largest open subscheme of $\operatorname{Spec} B$ over which $\Omega_{B/A}$ is locally free is therefore $(\operatorname{Spec} B) \smallsetminus \{\mathfrak{m}\}$.

(b) Let $J := (XY - ZW) \subset B' := A[X, Y, Z, W]$; then the second exact sequence $J/J^2 \to B \otimes_{B'} \Omega_{B'/A} \to \Omega_{B/A} \to 0$ yields an exact sequence of B-modules

$$B \longrightarrow B \cdot dX \oplus B \cdot dY \oplus B \cdot dZ \oplus B \cdot dW \longrightarrow \Omega_{B/A} \longrightarrow 0$$

where the homomorphism on the left sends 1 to $d(XY - ZW) = Y \cdot dX + X \cdot dY - W \cdot dZ - Z \cdot dW$. By a similar calculation as in (a) the image becomes a direct summand of rank 1 after inverting any one of X, Y, Z, W, but the homomorphism becomes zero modulo the maximal ideal $\mathfrak{m} := (X, Y, Z, W)$. Thus $\Omega_{B/A}$ is locally free of rank 3 over $(\operatorname{Spec} B) \smallsetminus \{\mathfrak{m}\}$, but not at $\mathfrak{m}$ itself.

(c) Let $J := (X^n) \subset B' := A[X]$; then the second exact sequence $J/J^2 \to B \otimes_{B'} \Omega_{B'/A} \to \Omega_{B/A} \to 0$ yields an exact sequence of B-modules

$$B \longrightarrow B \cdot dX \longrightarrow \Omega_{B/A} \longrightarrow 0$$

where the homomorphism on the left sends 1 to $d(X^n) = n \cdot X^{n-1} \cdot dX$. Thus $\Omega_{B/A}$ is isomorphic to the B-module $B/nX^{n-1}B$. If $n = 1$, this is zero and hence free. If k has characteristic $p > 0$ which divides n, we have $n = 0$ in k and hence $\Omega_{B/A}$ free of rank 1. If $n > 1$ is not a multiple of the characteristic of k, then $\Omega_{B/A} \cong B/X^{n-1}B$ is a module of length $n - 1$, while B itself has length n; hence $\Omega_{B/A}$ is not free. As B is already a local ring, we find that $\Omega_{B/A}$ is not locally free anywhere in that case.

6. Let k be perfect of characteristic $p > 0$. Let K be an extension of k. Show that $k \subset K^p$ and that the canonical homomorphism $\Omega^1_{K/k} \to \Omega^1_{K/K^p}$ is an isomorphism.

   Solution: Since k is perfect, every element of k is a p-th power of an element of k; hence $k \subset K^p$. Now consider any derivation $d \colon K \to V$ over $\mathbb{Z}$. Then for every $x \in K$ the Leibniz rule implies that $d(x^p) = px^{p-1}dx = 0$. It follows that every derivation of K over $\mathbb{Z}$ is already a derivation over K^p and hence over k. Thus $\Omega^1_{K/\mathbb{Z}}$ and $\Omega^1_{K/k}$ and Ω^1_{K/K^p} satisfy the same universal property, and so the canonical homomorphisms between them are isomorphisms.

7. Construct an inseparable field extension L/K with $\Omega_{L/K} = 0$.

Solution: Take $K := \mathbb{F}_p$, let X be a variable over K, and let L be the extension of K that is generated by the elements $X_i := X^{p^{-i}}$ for all $i \geqslant 0$. Let $d\colon L \to \Omega_{L/K}$ be the universal derivation over K. Then for every i we have $X_i = X_{i+1}^p$; hence the Leibniz rule implies that $dX_i = p \cdot X_{i+1}^{p-1} \cdot dX_{i+1} = 0$. This implies that $df = 0$ for any polynomial f in $X_0, X_1, \ldots$. Using the quotient rule we deduce that $d(\frac{f}{g}) = 0$ for any $\frac{f}{g} \in L$. Thus the whole map d is zero. Since its image generates the L-module $\Omega_{L/K}$, it follows that $\Omega_{L/K} = 0$, as desired.