D-MATH | Algebraic Geometry II | HS 2017
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Solutions 8

Sheaves of Differentials, Canonical Sheaf, Smoothness

Let k be a field. Recall that a *variety* over k is a reduced scheme X of finite type over k. We say that X is *nonsingular* if it is regular at every point.

1. Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on X. Show that any point $x \in X$, such that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, possesses an open neighborhood $U \subset X$ such that $\mathcal{F}|_U$ is free. Deduce that $\mathcal{F}$ is locally free if and only if $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all closed points $x \in X$.

   Solution: For the first statement, we may assume without loss of generality that $X = \operatorname{Spec} R$ is affine and that $\mathcal{F} = \tilde{M}$ for some finitely generated R-module M. Write $x = \mathfrak{p} \in \operatorname{Spec} R$. By assumption, there is an isomorphism $\varphi_\mathfrak{p} \colon R^n_\mathfrak{p} \xrightarrow{\sim} M_\mathfrak{p}$. Localizing if necessary, this extends to a homomorphism $\varphi\colon R^n \to M$. Then $\ker\varphi$ and $\operatorname{coker}\varphi$ are finitely generated R-modules with $(\ker\varphi)_\mathfrak{p} = (\operatorname{coker}\varphi)_\mathfrak{p} = 0$. Choose finitely many generators n_i and for each i choose $u_i \in R \smallsetminus \mathfrak{p}$ such that $\frac{n_i}{u_i} = 0$. Since $\mathfrak{p}$ is a prime ideal, the product u of these u_i then again lies in $R \smallsetminus \mathfrak{p}$ and satisfies $\frac{n_i}{u} = 0$ for all i. Thus $(\ker\varphi)_u = (\operatorname{coker}\varphi)_u = 0$, and by exactness of localization φ induces an isomorphism $R^n_u \xrightarrow{\sim} M_u$. Thus $\operatorname{Spec} R_u$ is an open neighborhood with the desired property.

   Since X is noetherian, every point in X specializes to a closed point. See [Stacks, Tag 01OU, Lemma 27.5.9]. Let $y \in X$ and let $x \in X$ be a closed point with $x \in \overline{\{y\}}$, then $\mathcal{F}_y \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,y}$. Thus the first statement implies the second.

2. (a) Let Y_1 and Y_2 be schemes over X and let $Y := Y_1 \times_X Y_2$. Construct a natural isomorphism
$$\Omega_{Y/S} \cong \mathrm{pr}_1^*\Omega_{Y_1/X} \oplus \mathrm{pr}_2^*\Omega_{Y_2/X}.$$

   (b) If Y_1 and Y_2 are nonsingular varieties over a perfect field k, construct a natural isomorphism
$$\omega_{Y/k} \cong \mathrm{pr}_1^*\omega_{Y_1/k} \otimes \mathrm{pr}_2^*\omega_{Y_2/k}.$$

   Solution: (a) By the base change property for differentials we have $\Omega_{Y/Y_1} \cong \mathrm{pr}_2^*\Omega_{Y_2/X}$. Combining this with the exact sequence associated to the composition $Y \to Y_1 \to X$ we obtain an exact sequence
$$\mathrm{pr}_1^*\Omega_{Y_1/X} \xrightarrow{i} \Omega_{Y/X} \longrightarrow \mathrm{pr}_2^*\Omega_{Y_2/X} \longrightarrow 0.$$

   We obtain a similar exact sequence by symmetry. In particular, we have a surjective morphism $j\colon \Omega_{Y/X} \twoheadrightarrow \mathrm{pr}_1^*\Omega_{Y_1/X}$. We claim that $j \circ i = \mathrm{id}_{\mathrm{pr}_1^*\Omega_{Y_1/X}}$. It suffices to prove this when X and Y_i for $i = 1, 2$ are affine, in which case the desired result follows easily from writing the relevant maps explicitly.

(b) Let $n := \dim Y$ and $n_i := \dim Y_i$ for $i = 1, 2$. Since all sheaves involved are locally free, the same proof as in linear algebra yields the identity

$$\bigwedge^n (\mathrm{pr}_1^* \Omega_{Y_1/X} \oplus \mathrm{pr}_2^* \Omega_{Y_2/X}) \cong \bigoplus_{p+q=n} \Big(\bigwedge^p \mathrm{pr}_1^* \Omega_{Y_1/X} \otimes \bigwedge^q \mathrm{pr}_2^* \Omega_{Y_2/X}\Big).$$

Since $\Omega_{Y_i/X}$ is locally free of rank n_i for $i = 1, 2$, the only non-trivial term on the right-hand side is

$$\bigwedge^{n_1} \mathrm{pr}_1^* \Omega_{Y_1/X} \otimes \bigwedge^{n_2} \mathrm{pr}_2^* \Omega_{Y_2/X} = \mathrm{pr}_1^* \omega_{Y_1/k} \otimes \mathrm{pr}_2^* \omega_{Y_2/k}.$$

We thus obtain the desired result by applying $\bigwedge^n$ to both sides of the isomorphism from part (a).

3. Let X be a nonsingular variety over an algebraically closed field k. We call $\mathcal{T}_X := \mathscr{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ the *(relative) tangent sheaf of X (over k).* A global section of $\mathcal{T}_X$ is called a *tangent field on X.*

   (a) Show that $\mathcal{T}_X$ is locally free. What is its rank?

   (b) Describe $\mathcal{T}_{\mathbb{P}^n_k}$ by an explicit short exact sequence.

   (c) Does $\mathbb{P}^1_k$ possess a nowhere vanishing tangent field?

   **(d) Does $\mathbb{P}^n_k$ possess a nowhere vanishing tangent field for arbitrary n?

   Solution: (a) Since $\Omega_{X/k}$ is locally free of rank $n := \dim X$ and $\mathcal{T}_X = \Omega^\vee_{X/k}$, it follows directly that $\mathcal{T}_X$ is locally free of rank n as well.

   (b) Recall that there is a short exact sequence

$$0 \to \Omega_{\mathbb{P}^n_k/k} \to \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n_k} \to 0.$$

   On applying the left exact functor $\mathscr{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ we obtain a sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_{\mathbb{P}^n_k}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n_k} \to 0$$

   which is exact except possibly on the right. But from commutative algebra we know that any short exact sequence of modules $0 \to M \to M'' \to M' \to 0$ over a ring A splits if M' is a free A-module. Moreover, the functor $\mathrm{Hom}_A(-, A)$ preserves split exact sequences. Applying this to any open affine $\mathrm{Spec}\, A \subset \mathbb{P}^n_k$ we deduce that the sequence is also right exact.

   (c) In this case we have $\Omega_{\mathbb{P}^1_k/k} \cong \omega_{\mathbb{P}^1/k} \cong \mathcal{O}_{\mathbb{P}^1_k}(-2)$. It follows that $\mathcal{T}_{\mathbb{P}^n_k} \cong \mathcal{O}_{\mathbb{P}^1_k}(2)$. A tangent field thus corresponds to a homogeneous polynomial of degree 2 on $\mathbb{P}^1_k$. Since any such polynomial has a zero in $\mathbb{P}^1_k$, it follows that every tangent field on $\mathbb{P}^1_k$ must vanish somewhere.

*4. Let $i\colon Y \hookrightarrow X$ be a closed immersion of codimension 1 of a nonsingular variety X over an algebraically closed field k, whose ideal sheaf $\mathcal{J}$ can be locally generated by one element at every point. We define the *canonical sheaf* of such Y as

$$\omega^\circ_{Y/k} := i^*\omega_{X/k} \otimes i^*(\mathcal{J}/\mathcal{J}^2)^\vee.$$

(a) Prove that $\omega^\circ_{Y/k}$ is an invertible sheaf.

(b) Verify that $\omega^\circ_{Y/k} \cong \omega_{Y/k}$ if Y is nonsingular.

(c) Determine $\Omega_{Y/k}$ and $\omega^\circ_{Y/k}$ for the nodal curve $Y = V(C(C-B)A - B^3) \subset \mathbb{P}^2_k$ and explain the difference.

Solution: (a) Since X is nonsingular, its local rings are regular and hence integral. Thus $\mathcal{J}$ is locally generated by one element in an integral domain and so locally free of rank 1, in other words, an invertible sheaf on X. Thus $i^*(\mathcal{J}/\mathcal{J}^2)$ is an invertible sheaf on Y. Since $\omega_{X/k}$ is invertible and tensor products, pullbacks and duals of invertible sheaves are also invertible, the result follows.

(b) Tensoring both sides of the natural isomorphism $i^*\omega_{X/k} \cong \omega_{Y/k} \otimes i^*(\mathcal{J}/\mathcal{J}^2)$ from the course by $i^*(\mathcal{J}/\mathcal{J}^2)^\vee$ yields an isomorphism

$$\omega_{Y/k} \cong i^*\omega_{X/k} \otimes i^*(\mathcal{J}/\mathcal{J}^2)^\vee = \omega^\circ_{Y/k}$$

as desired.

(c) In this case $\omega_{\mathbb{P}^2_k/k} \cong \mathcal{O}_{\mathbb{P}^2_k}(-3) \cong \mathcal{J}$ because the curve has degree 3. Therefore $\omega^\circ_{Y/k} \cong \mathcal{O}_Y$. Also Y is non-singular except at the point $y := (1:0:0)$; so by (b) the sheaf $\omega^\circ_{Y/k}$ is naturally isomorphic to $\omega_{Y/k} = \Omega_{Y/k}$ over $Y \smallsetminus \{y\}$.

To determine $\Omega_{Y/k}$ near y we look at the affine chart $D_A \subset \mathbb{P}^2_k$. Using the coordinates $b := \frac{B}{A}$ and $c := \frac{C}{A}$ we have $Y \cap D_A = \operatorname{Spec} R$ for $R := k[b,c]/(f)$ with the polynomial $f(b,c) := c(c-b) - b^3$. The second exact sequence for differentials yields the presentation

$$\begin{array}{ccccccc}
(\mathcal{J}/\mathcal{J}^2)(Y\cap D_A) & \xrightarrow{\ d\ } & (i^*\Omega_{X/k})(Y\cap D_A) & \longrightarrow & \Omega_{Y/k}(Y\cap D_A) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
R\cdot[f] & \xrightarrow{\ d\ } & R\cdot db \oplus R\cdot dc & \longrightarrow & \Omega_{R/k} & \longrightarrow & 0. \\
[f] & \longmapsto & [df] = [\frac{\partial f}{\partial b}]\cdot db + [\frac{\partial f}{\partial c}]\cdot dc & & & &
\end{array}$$

To read off the structure of $\Omega_{R/k}$, observe that the homomorphism

$$\pi\colon\ R\cdot db \oplus R\cdot dc \longrightarrow R, \quad g\cdot db + h\cdot dc \mapsto g\cdot[\tfrac{\partial f}{\partial c}] - h\cdot[\tfrac{\partial f}{\partial b}]$$

is zero on the image of d. Also the polynomials $\frac{\partial f}{\partial b} = -c - 3b^2$ and $\frac{\partial f}{\partial c} = 2c - b$ and f together generate the maximal ideal $(b,c) \subset k[b,c]$. Thus the image of π

is the maximal ideal $\mathfrak{m} := (b,c)/(f)$ of R. Another little computation shows that $\ker \pi = \operatorname{im}(d)$; hence π induces an isomorphism $\Omega_{R/k} \xrightarrow{\sim} \mathfrak{m}$. In particular we see that $\Omega_{Y/k}$ is not locally free at y. (Note: It is easy to see that d is injective here, but that does not matter for the stated question.)

To describe $\Omega_{Y/k}$ globally and to compare it with $\omega^\circ_{Y/k}$, observe that the above formula for π appears canonically in the natural pairing

$$\begin{aligned}(R\cdot db \oplus R\cdot dc) \times R\cdot [f] &\longrightarrow R\cdot db\wedge dc,\\ (\omega,[f]) &\longmapsto \omega\wedge[df] = \pi(\omega)\cdot db\wedge dc.\end{aligned}$$

This in turn arises by taking sections over $Y\cap D_A$ from the natural pairing

$$\begin{aligned} i^*\Omega_{\mathbb{P}^2_k/k} \times i^*(\mathcal{J}/\mathcal{J}^2) &\longrightarrow i^*\Omega^2_{\mathbb{P}^2_k/k} = i^*\omega_{\mathbb{P}^2_k/k},\\ (\omega,[g]) &\longmapsto \omega\wedge[dg].\end{aligned}$$

As the latter induces the isomorphism $\omega_{Y/k}\cong\omega^\circ_{Y/k}$ outside y, by this pairing we can identify $\omega^\circ_{Y/k}$ naturally with the free module of rank 1 with basis $\frac{db\wedge dc}{[df]}$ over $Y\cap D_A$. The above calculation for π then yields an isomorphism between $\Omega_{Y/k}$ and $\mathfrak{m}\cdot\frac{db\wedge dc}{[df]}$ over $Y\cap D_A$. Together this yields a natural isomorphism

$$\Omega_{Y/k} \;\cong\; \mathfrak{M}\cdot\omega^\circ_{Y/k} \;\cong\; \mathfrak{M}\otimes\omega^\circ_{Y/k},$$

where $\mathfrak{M}\subset\mathcal{O}_Y$ is the ideal sheaf of the singular point y. Since $\omega^\circ_{Y/k}\cong\mathcal{O}_{Y/k}$, we deduce that $\Omega_{Y/k}\cong\mathfrak{M}$.

To summarize, $\omega^\circ_{Y/k}$ and $\Omega_{Y/k}$ are isomorphic where Y is regular, but it was clear in advance that they cannot be isomorphic at the singular point y, because the former is an invertible sheaf by (a), but the latter is not locally free there because of the isomorphism $\Omega_{Y/k}\otimes_{\mathcal{O}_{Y,y}} k(y)\cong\mathfrak{m}/\mathfrak{m}^2$. (We will see later that $\omega^\circ_{Y/k}$ really does play the same important role that is played by $\omega_{Y/k}$ in the regular case.)

5. (*Smooth Morphisms and the Jacobian Criterion*) Let $f\colon X\to Y$ be a morphism of schemes and $d\geqslant 0$ an integer. We say that f is *smooth of relative dimension d* at $x\in X$, if there exist affine open neighborhoods U of x and $V=\operatorname{Spec} R$ of $f(x)$ such that $f(U)\subset V$, and an open immersion

$$j\colon U\hookrightarrow \operatorname{Spec} R[T_1,\ldots,T_n]/(f_1,\ldots,f_{n-d})$$

of R-schemes for suitable n and f_i, such that the Jacobian matrix

$$J_{f_1,\ldots,f_{n-d}}(x) := \left(\frac{\partial f_i}{\partial T_j(x)}\right)_{i,j} \in M_{n-d\times n}(\kappa(x))$$

has rank $n-d$. We call f *smooth* if it is smooth at all points $x\in X$. Show:

(a) Smoothness is local on the source and the target.

(b) Smoothness is invariant under base change.

(c) Smoothness is invariant under composition.

(d) Every open immersion is smooth of relative dimension 0.

(e) The set of points of X at which f is smooth is open.

Solution: (a), (b) and (d) are clear from the definition.

(c) Suppose that $f\colon X \to Y$ is smooth of relative dimension d at $x \in X$ and $g\colon Y \to Z$ is smooth of relative dimension e at $f(x) \in Y$. Then there are affine open neighborhoods U of x and $f(U) \subset V = \operatorname{Spec} A$ of $f(x)$ and $g(V) \subset W = \operatorname{Spec} B$ with open immersions

$$j\colon U \hookrightarrow \operatorname{Spec} A[S_1, \dots, S_n]/(f_1, \dots, f_{n-d})$$

and

$$k\colon V \hookrightarrow \operatorname{Spec} B[T_1, \dots, T_m]/(g_1, \dots, g_{m-e}).$$

Let $F := (f_1, \dots, f_{n-d}) \subset A[\underline{S}]$ and $G := (g_1, \dots g_{m-e}) \subset B[\underline{T}]$ and let $C := B[\underline{T}]/G$. Since k is an open immersion, there is a standard affine open neighborhood of $f(x)$ of the form $\operatorname{Spec} C_h \subset V$. Note that $C_h \cong C[T]/(hT-1)$. Letting $g_{m+1-e} := hT - 1$, the Jacobian $J_{g_1,\dots,g_{m+1-e}}\big(f(x)\big)$ has rank $m+1-e$ since

$$h\big(f(x)\big) = \frac{\partial g_{m+1-e}}{\partial T} \neq 0.$$

By replacing V with $\operatorname{Spec} C_h$ and U with an affine open neighborhood of x contained in $f^{-1}(\operatorname{Spec} C_h)$, we may thus assume that k is an isomorphism. But then the composition

$$U \hookrightarrow \operatorname{Spec} A[\underline{S}]/F \cong \operatorname{Spec} C[\underline{S}]/F \cong \operatorname{Spec} B[\underline{T}, \underline{S}]/F + G$$

is an open immersion. The corresponding Jacobian is a block matrix of the form

$$\begin{pmatrix} J_{g_1,\dots,g_{m-e}}(x) & * \\ 0 & J_{f_1,\dots,f_{n-d}}(x) \end{pmatrix},$$

which has rank $n+m-(d+e)$. It follows that $g \circ f$ is smooth of relative dimension $d+e$ at x. This also implies that if f and g are smooth morphisms, then $g \circ f$ is smooth.

(e) The rank condition in the definition of smoothness can be phrased by saying that there exists an $r \times r$ minor of the Jacobian matrix which does not vanish at x. If such a minor does not vanish at x, it also does not vanish in a neighborhood of x. This means that if f is smooth of relative dimension d at x, then it is also smooth of relative dimension d in a neighborhood of x; hence the set of points of X at which f is smooth is open.

6. Let X be a scheme of finite type over k, where k is perfect. We call X *smooth* over k if the structure morphism $X \to \operatorname{Spec} k$ is smooth. Assume that X is irreducible, and show that X is smooth over k if and only if $\Omega^1_{X/k}$ is locally free of rank $\dim X$.

Solution: We separate the proof into steps:

(1) As a preparation consider an ideal $I := (f_1, \ldots, f_m) \subset k[X_1, \ldots X_n]$ and let $R := k[\underline{X}]/I$. The second exact sequence for the surjection $k[\underline{X}] \twoheadrightarrow R$ reads

$$I/I^2 \xrightarrow{d} R \otimes_{k[\underline{X}]} \Omega_{k[\underline{X}]/k} \longrightarrow \Omega_{R/k} \longrightarrow 0.$$

Here the term in the middle is a free R-module with basis $dX_1, \ldots, dX_n$. Also I/I^2 is generated by the residue classes of $f_1, \ldots, f_m$, and for each i we have $df_i = \sum_j \frac{\partial f_i}{X_j} dX_j$. Thus we find an exact sequence

$$(*) \qquad R^m \xrightarrow{J} R^n \longrightarrow \Omega_{R/k} \longrightarrow 0$$

with the jacobian matrix $J := J_{f_1,\ldots,f_{n-d}} := \big(\frac{\partial f_i}{X_j}\big)_{i,j}$.

(2) Now suppose that X is smooth of relative dimension d over k. Then for any closed point $x \in X$ there exist an open affine neighborhood U of x and an open immersion $j \colon U \hookrightarrow \operatorname{Spec} R$ for

$$R := k[T_1, \ldots, T_n]/(f_1, \ldots, f_{n-d}),$$

such that $J_{f_1,\ldots,f_{n-d}}(x)$ is an $n \times (n-d)$-matrix of rank $n-d$ at x. After possibly permuting $X_1, \ldots, X_n$ we may assume that the submatrix $(\frac{\partial f_i}{X_j})_{i,j=1}^{n-d}$ is invertible at x. Let $\mathfrak{m} \subset R$ be the maximal ideal associated to x. Then the composite homomorphism $R_\mathfrak{m}^{n-d} \xrightarrow{J} R_\mathfrak{m}^n \xrightarrow{\pi} R_\mathfrak{m}^{n-d}$, where π denotes the projection to the first $n-d$ variables, is an isomorphism. The inclusion $R_\mathfrak{m}^d \hookrightarrow R_\mathfrak{m}^n$, $\underline{y} \mapsto (\underline{0}, \underline{y})$ thus induces an isomorphism

$$R_\mathfrak{m}^d \xrightarrow{\sim} \operatorname{coker}\big(J \colon R_\mathfrak{m}^{n-d} \to R_\mathfrak{m}^n\big) \overset{(*)}{\cong} \Omega_{R/k} \otimes_R R_\mathfrak{m}.$$

Since x was an arbitrary closed point, by exercise 1 it follows that $\Omega_{X/k}$ is locally free of rank d.

(3) In the situation of (2) we claim that $d = \dim X$, so that a scheme which is "smooth of relative dimension d over k" is actually of dimension d. To show this consider any closed point $x \in X$, corresponding to the maximal ideal $\mathfrak{m} \subset R$. Then the residue field $k(x) = R/\mathfrak{m}$ is a finite extension of k. Since k is perfect, this extension is separable. By the course we therefore have a natural isomorphism $\Omega_{R/k} \otimes_R k(x) \cong \mathfrak{m}/\mathfrak{m}^2$. Since $\Omega_{R/k}$ is locally free of rank d, it follows that $\dim_{k(x)} \mathfrak{m}/\mathfrak{m}^2 = d$. Thus $\dim \mathcal{O}_{X,x} \leqslant d$. On the other hand the ideal of X is generated by $n-d$ elements, so by the Krull dimension theorem we have $\operatorname{codim}_{X \subset \mathbb{A}^n_k} \leqslant n-d$ and hence $\dim X \geqslant d$. Together it follows that $\dim X = \dim \mathcal{O}_{X,x} = \dim_{k(x)} \mathfrak{m}/\mathfrak{m}^2 = d$.

(4) Conversely suppose that $\Omega_{X/k}$ is locally free of rank $d := \dim(X)$. Since smoothness is local, we may assume that $X = \operatorname{Spec} R$ with $R = k[X_1, \ldots, X_n]/I$ and $I = (f_1, \ldots f_m)$. Let $x \in X$ be a closed point corresponding to a maximal ideal $\mathfrak{m} \subset R$. Then the exact sequence $(*)$ implies that $J_{f_1,\ldots,f_m}(x)$ has rank $n-d$. Choose $n-d$ of the f_i corresponding to an invertible $(n-d)\times(n-d)$ minor. After a possible permutation we may assume that these are $f_1, \ldots, f_{n-d}$. Let $X' \subset \mathbb{A}_k^n$ be the closed subscheme defined by the ideal $(f_1, \ldots f_{n-d})$. By the same argument as in (c) we have $\dim \mathcal{O}_{X',x} = \dim_{k(x)} \mathfrak{m}'/\mathfrak{m}'^2 = d$, where $\mathfrak{m}'$ is the maximal ideal of $\mathcal{O}_{X',x}$. Thus $\mathcal{O}_{X',x}$ is a regular local ring. In particular it is an integral domain. As it surjects to $\mathcal{O}_{X,x}$, which by assumption has the same Krull dimension d, the surjection must be an isomorphism. It follows that the embedding $X \hookrightarrow X'$ is an isomorphism at x and hence over a whole neighborhood of x. By construction X' is smooth of relative dimension d over k at x; hence so is X. Since $x \in X$ was an arbitrary closed point and the set of closed points of X is dense and smoothness is an open condition, we conclude that f is smooth.

(5) *Remark:* The same statement also holds if k is not perfect. The main point is to show that in either case the set of closed points of X whose residue fields are separable extensions of k is Zariski dense.

For this see https://stacks.math.columbia.edu/tag/056U.