Algebraic Geometry II

Solutions 9

Čech Cohomology

1. Show that the complex of abelian groups $\ldots \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \ldots$ is acyclic but not contractible.

   Solution: Let us denote the complex by $C^\bullet$. It is clear that $C^\bullet$ is exact and hence acyclic. Suppose $C^\bullet$ is contractible. Then there exists a degree -1 morphism of complexes $h\colon C^\bullet \to C^\bullet$ with $\mathrm{id} = 2 \circ h + h \circ 2$. In each degree, the images of both $2 \circ h$ and $h \circ 2$ must lie in $2\mathbb{Z}/4\mathbb{Z}$. But two submodules of $2\mathbb{Z}/4\mathbb{Z}$ can never generate $\mathbb{Z}/4\mathbb{Z}$. Hence the identity cannot be written as such a sum, a contradiction. Thus $C^\bullet$ is not contractible.

2. Let X be a separated and quasicompact scheme and let $(\mathcal{F}_i)_{i\in I}$ be a filtered direct system of quasicoherent sheaves on X. Show that for any $p \geqslant 0$ there is a natural isomorphism

   $$\varinjlim_i H^p(X, \mathcal{F}_i) \cong H^p(X, \varinjlim_i \mathcal{F}_i).$$

   Solution: We begin with some preparations that tell us what the direct limit does with quasicoherent sheaves on an arbitrary scheme X.

   Claim 1. *Let $(\mathcal{F}_i)_{i\in I}$ be a direct system of sheaves on X. Then for any open subscheme $U \subset X$ there is a natural isomorphism*

   $$(\varinjlim_i \mathcal{F}_i)|_U \cong \varinjlim_i (\mathcal{F}_i|_U).$$

   Proof. Let $j\colon U \hookrightarrow X$ denote the canonical open inclusion. Then for any sheaf $\mathcal{F}$ on X we have $\mathcal{F}|_U = j^{-1}\mathcal{F}$. Recall that j^{-1} is left-adjoint to the functor j_* from the category of sheaves on U to the category of sheaves on X. It follows that j^{-1} commutes with direct limits, as desired. □

   Claim 2. *Let $X = \operatorname{Spec} A$ and let $(M_i)_i$ be a direct system of A-modules. Then there is a natural isomorphism of quasi-coherent sheaves*

   $$(\varinjlim_i M_i)^\sim \cong \varinjlim_i \tilde{M}_i.$$

   Proof. By §5.3 of the course the functor $M \mapsto \tilde{M}$ is left adjoint to the global sections functor; hence it commutes with direct limits. □

Claim 3. *For any direct system of quasi-coherent sheaves on X the sheaf $\varinjlim_i \mathcal{F}_i$ is quasi-coherent, and for any open affine $U \subset X$ there is a natural isomorphism*

$$(\varinjlim_i \mathcal{F}_i)(U) \cong \varinjlim_i (\mathcal{F}_i(U)).$$

Proof. Combine Claims 1 and 2. □

Now assume that X is separated and quasicompact and that $(\mathcal{F}_i)_{i\in I}$ is a filtered direct system of quasi-coherent sheaves on X. Then $\varinjlim_i \mathcal{F}_i$ is also quasi-coherent, so we may compute both sides of the desired isomorphism with a fixed finite open affine covering $\mathcal{U} = (U_i)_{i=1,\ldots,n}$ of X. Since X is separated, each intersection $U_{i_0\ldots i_p} := U_{i_0} \cap \ldots \cap U_{i_p}$ is then again affine. By Claim 3 and the fact that direct limits of abelian groups commute with finite products we deduce that for each $p \geqslant 0$ we have a natural isomorphism

$$\varinjlim_i C^p(\mathcal{U}, \mathcal{F}_i) \;=\; \varinjlim_i \prod_{i_0,\ldots,i_p} \mathcal{F}_i(U_{i_0\ldots i_p}) \;\cong\; \prod_{i_0,\ldots,i_p} \big(\varinjlim_i \mathcal{F}_i\big)(U_{i_0\ldots i_p}) \;=\; C^p(\mathcal{U}, \varinjlim_i \mathcal{F}_i).$$

The naturality also implies that these isomorphisms combine to an isomorphism of complexes

$$\varinjlim_i C^\bullet(\mathcal{U}, \mathcal{F}_i) \;\cong\; C^\bullet(\mathcal{U}, \varinjlim_i \mathcal{F}_i).$$

Now note that in the category of modules over a ring, taking the direct limit over a filtered set is an exact functor. For this, see [Stacks, Tag 00DB, Lemma 10.8.8]. This means that taking filtered direct limits commutes with kernels and cokernels. Thus we obtain the desired isomorphism

$$\varinjlim_i H^p(X, \mathcal{F}_i) \;\cong\; \varinjlim_i H^p(\mathcal{U}, \mathcal{F}_i) \;\cong\; H^p(\mathcal{U}, \varinjlim_i \mathcal{F}_i) \;\cong\; H^p(X, \varinjlim_i \mathcal{F}_i).$$

3. Let X be a scheme.

   (a) Construct a natural isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$.

   *(b) Suppose that X is integral and let $\mathcal{K}_X$ denote the constant sheaf with values in $K(X)$. Show that the exact sequence

   $$1 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 1$$

   induces the isomorphism $\mathrm{DivCl}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$ from §5.9 of the course.

Solution: (a) We divide the solution into steps:

(1) Let $\mathcal{L}$ be an invertible sheaf on X. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X such that for every $i \in I$, the restriction $\mathcal{L}|_{U_i}$ is free, generated by a section $e_i \in \mathcal{L}(U_i)$. For every i, j, the $e_i|_{U_{ij}}$ and $e_j|_{U_{ij}}$ are both generators of $\mathcal{L}|_{U_{ij}}$, and

hence differ by an automorphism of $\mathcal{O}_{U_{ij}}$. There thus exist $f_{ij} \in \mathcal{O}_X(U_{ij})^\times$ such that $e_i|_{U_{ij}} = f_{ij} \cdot e_j|_{U_{ij}}$. Moreover, for every i, j, k we have

$$f_{ik}|_{U_{ijk}} \cdot e_k|_{U_{ijk}} = e_i|_{U_{ijk}} = f_{ij}|_{U_{ijk}} \cdot e_j|_{U_{ijk}} = (f_{ij}|_{U_{ijk}} \cdot f_{jk}|_{U_{ijk}}) \cdot e_k|_{U_{ijk}},$$

and hence

$$f_{ij}|_{U_{ijk}} \cdot f_{jk}|_{U_{ijk}} = f_{ik}|_{U_{ijk}}. \tag{$*$}$$

This means that the 1-cochain $f := (f_{ij})_{ij} \in C^1(\mathcal{U}, \mathcal{O}_X^\times)$ satisfies

$$df = \big(f_{jk}|_{U_{ijk}} \cdot f_{ik}^{-1}|_{U_{ijk}} \cdot f_{ij}|_{U_{ijk}}\big)_{ijk} = (1)_{ijk}$$

and so is a 1-cocyle.

(2) Any different choice of generators of $\mathcal{L}|_{U_{ij}}$ has the form $g_i e_i$ for sections $e_i \in \mathcal{O}_X(U_i)^\times$ and results in the 1-cocycle $(g_i|_{U_{ij}} \cdot g_j^{-1}|_{U_{ij}} \cdot f_{ij})_{ij}$. This differs from f by the coboundary $d(g_j^{-1})$. Thus the class $[f] \in H^1(\mathcal{U}, \mathcal{O}_X^\times)$ is independent of the choice of the e_i.

(3) Let $\mathcal{V} = (V_j)_{j \in J}$ be a refinement of $\mathcal{U}$, with $\sigma\colon J \to I$ such that $V_j \subset U_{\sigma(j)}$ for every $j \in J$. Then for each j the restriction $e_{\sigma(j)}|_{V_j}$ is a generator for $\mathcal{L}|_{V_j}$. The cocycle in $C^1(\mathcal{V}, \mathcal{O}_X^\times)$ associated to these is simply $\sigma^* f$. It follows that the image of $[f]$ in $H^1(X, \mathcal{O}_X^\times) = \varinjlim_{\mathcal{V}} H^1(\mathcal{V}, \mathcal{O}_X^\times)$ is independent of the choice of $\mathcal{U}$. We denote it by $\varphi(\mathcal{L})$. Clearly it depends only on the isomorphism class of $\mathcal{L}$, so this defines a map

$$\varphi\colon \operatorname{Pic}(X) \longrightarrow H^1(X, \mathcal{O}_X^\times).$$

(4) For two invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$ on X, choose an open covering $\mathcal{U}$ which trivializes both. For every i, let e_i and e_i' be generators of $\mathcal{L}|_{U_i}$ and $\mathcal{L}'|_{U_i}$, respectively. Define f and f' as in step (1) for $\mathcal{L}$ and $\mathcal{L}'$. Then each $e_i \otimes e_i'$ is a generator of $(\mathcal{L} \otimes \mathcal{L}')|_{U_i}$ and the associated 1-cocycle is $(f_{ij} \cdot f_{ij}')_{ij}$. It follows that $\varphi(\mathcal{L} \otimes \mathcal{L}')$ is equal to the class of $f \cdot f'$ and hence equal to $\varphi(\mathcal{L}) \cdot \varphi(\mathcal{L}')$. Thus the map φ is a homomorphism.

(5) If $\varphi(\mathcal{L}) = 1$, we have $[\sigma^* f] = [1]$ in $H^1(\mathcal{V}, \mathcal{O}_X^\times)$ for some refinement $\mathcal{V}$ of $\mathcal{U}$ and some σ as in (3). After replacing $\mathcal{U}$ by $\mathcal{V}$ we may assume that $[f] = [1]$ in $H^1(\mathcal{U}, \mathcal{O}_X^\times)$. This means that $f = dg$ for a 0-cocycle $g = (g_i)_i \in C^0(\mathcal{U}, \mathcal{O}_X^\times)$, in other words that $f_{ij} = g_j|_{U_{ij}} \cdot g_i^{-1}|_{U_{ij}}$ for all i, j. Then $e_i|_{U_{ij}} = g_j|_{U_{ij}} \cdot g_i^{-1}|_{U_{ij}} \cdot e_j|_{U_{ij}}$ and hence $g_i e_i|_{U_{ij}} = g_j e_j|_{U_{ij}}$. Thus the sections $g_i e_i$ glue to a global section $e' \in \mathcal{L}(X)$. As they generate $\mathcal{L}$ over each U_i, this yields an isomorphism $\mathcal{O}_X \xrightarrow{\sim} \mathcal{L}$. It follows that the homomorphism φ is injective.

(6) For surjectivity, let $c \in H^1(X, \mathcal{O}_X^\times)$. Then c is represented by some $[f] \in H^1(\mathcal{U}, \mathcal{O}_X^\times)$ for an open covering $\mathcal{U} = \{U_i\}_{i \in I}$. For every i, let $\mathcal{L}_i := \mathcal{O}_{U_i}$. Then multiplication by f_{ij} induces an isomorphism $\mathcal{L}_i|_{U_{ij}} \cong \mathcal{L}_j|_{U_{ij}}$ for every i, j. Since f is a 1-cocycle, these isomorphisms glue to an invertible sheaf $\mathcal{L}$ on X. (Compare

Section 3.2 of Spring 2017.) By construction, we see that $\varphi(\mathcal{L}) = c$. Hence φ is surjective, and we conclude that φ is an isomorphism.

(b) The relevant part of the associated long exact cohomology sequence, see [Liu, Prop 5.2.15] is

$$\ldots \to H^0(X, \mathcal{K}_X^\times) \xrightarrow{\psi} H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^\times) \to \ldots.$$

Here $H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ is the group of Cartier divisors and the image of ψ the subgroup of principal Cartier divisors; hence $\operatorname{coker}(\psi) \cong \operatorname{DivCl}(X)$. Any Cartier divisor D on X is determined by an open covering $\mathcal{U} = (U_i)_{i\in I}$ of X and sections $f_i \in \mathcal{K}_X^\times(U_i)$ such that for every i, j, we have $\frac{f_i|_{U_{ij}}}{f_j|_{U_{ij}}} \in \mathcal{O}_X^\times(U_{ij})$. The associated invertible sheaf $\mathcal{O}(D)$ is given by $\mathcal{O}(D)|_{U_i} = \mathcal{O}_{U_i} \cdot f_i^{-1} \subset \mathcal{K}|_{U_i}$ for each i. The 1-cocycle associated to $\mathcal{O}(D)$ in (a) is then $g := (\frac{f_j|_{U_{ij}}}{f_i|_{U_{ij}}})_{ij} \in C^1(\mathcal{U}, \mathcal{O}_X^\times)$. An explicit calculation using the snake lemma shows that $[g]$ is precisely the image of D under the connecting homomorphism δ. (Or is it the image of $-D$?)

4. Compute $H^*(X, \mathcal{O}_X)$ for $X = \mathbb{P}_k^2 \smallsetminus \{(0:0:1)\}$ and $X = \mathbb{A}_k^2 \smallsetminus \{(0,0)\}$ for a field k. Conclude that X is not affine.

   Solution: (a) Write $\mathbb{A}_k^2 = \operatorname{Spec} k[X,Y]$. Then $X = \mathbb{A}_k^2 \smallsetminus \{(0,0)\}$ has the affine open covering $\mathcal{U} = \{D_X, D_Y\}$. Since X is separated, it follows that $H^n(\mathcal{U}, \mathcal{O}_X) \cong H^n(X, \mathcal{O}_X)$. The ordered Čech complex (with $X < Y$) for $\mathcal{F}$ with respect to $\mathcal{U}$ reads

$$\begin{array}{c} \ldots \longrightarrow 0 \longrightarrow k[X^{\pm1}, Y] \times k[X, Y^{\pm1}] \longrightarrow k[X^{\pm1}, Y^{\pm1}] \longrightarrow 0 \longrightarrow \ldots \\ (f,g) \longmapsto g - f \end{array}$$

   and is non-zero only in degrees 0 and 1. Thus

$$H^n(X, \mathcal{O}_X) = \begin{cases} k[X,Y] & \text{if } n = 0, \\ k[X^{\pm1}, Y^{\pm1}]/(k[X^{\pm1}, Y] + k[X, Y^{\pm1}]) \cong \bigoplus_{i,j<0} k \cdot X^iY^j & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Now write $\mathbb{P}_k^2 = \operatorname{Proj} R$, where $R := k[X,Y,Z]$. Then $X := \mathbb{P}_k^2 \smallsetminus \{(0:0:1)\}$ has the affine open covering $\mathcal{U} = \{\overline{D}_X, \overline{D}_Y\}$, and so $H^n(\mathcal{U}, \mathcal{O}_X) \cong H^n(X, \mathcal{O}_X)$ by the same reasoning as above. The ordered Čech complex is

$$\begin{array}{c} \ldots \longrightarrow 0 \longrightarrow R_{X,0} \times R_{Y,0} \longrightarrow R_{XY,0} \longrightarrow 0 \longrightarrow \ldots \\ (f,g) \longmapsto g - f. \end{array}$$

   Thus

$$H^n(X, \mathcal{O}_X) = \begin{cases} R_{X,0} \cap R_{Y,0} = k & \text{if } n = 0, \\ R_{XY,0}/(R_{X,0} + R_{Y,0}) = \bigoplus_{i,j>0} k \cdot \frac{Z^{i+j}}{X^iY^j} & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where we take the intersection $R_{X,0} \cap R_{Y_0}$ inside of $R_{XY,0}$.

(c) In both cases, we have $H^1(X, \mathcal{O}_X) \neq 0$, and it follows from Serre's criterion that X is not affine. (Compare Spring 2017, Sheet 6, Exercise 5)

*5. Find a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on an affine scheme X for which $H^n(X, \mathcal{F}) \neq 0$ for some $n > 0$.

Solution: Let $X = \mathbb{A}^1_k = \operatorname{Spec} k[T]$ and $U := X \smallsetminus \{0\}$. Consider the canonical inclusion $j \colon U \hookrightarrow X$. Recall that we have an extension by zero sheaf $j_!\mathcal{O}_U$ on X, which has the structure of an $\mathcal{O}_X$-module but is **not** quasicoherent. Let $i \colon \{0\} \hookrightarrow \mathbb{A}^1_k$ be the inclusion of the origin into $\mathbb{A}^1$ (as topological spaces, not as schemes). By [Hartshorne, Exercise II.2.19], there is a short exact sequence

$$0 \to j_!\mathcal{O}_U \to \mathcal{O}_X \to i_*i^{-1}\mathcal{O}_X \to 0.$$

By [Liu, Proposition 5.2.15], we have an associated exact sequence

$$\ldots \to H^0(X, \mathcal{O}_X) \to H^0(X, i_*i^{-1}\mathcal{O}_X) \to H^1(X, j_!\mathcal{O}_U) \to H^1(X, \mathcal{O}_X) \to \ldots$$

Recalling that H^0 is naturally isomorphic to the global sections functor, we have $H^0(X, \mathcal{O}_X) \cong k[T]$ and $H^0(X, i_*i^{-1}\mathcal{O}_X) \cong H^0(\{0\}, i^{-1}\mathcal{O}_X) \cong \mathcal{O}_{X,0} \cong k[T]_{(T)}$ and the map between them is the natural inclusion. Moreover, since X is affine and $\mathcal{O}_X$ is quasi-coherent, we have $H^1(X, \mathcal{O}_X) = 0$. It follows that

$$H^1(X, j_!\mathcal{O}_U) \;\cong\; k[T]_{(T)}/k[T] \;\neq\; 0.$$