

EXERCISES : SHEET II.

Exercise 1

Prove that $p_c = \sup \{ p \in [0, 1] : P_p(N=0) = 1 \}$.

Exercise 2

An event A is called decreasing if A^c is increasing.

1) Let A, B be two decreasing events. Prove that

$$P_p(A \cap B) \geq P_p(A) P_p(B).$$

2) Let A increasing and B decreasing. Prove that.

$$P_p(A \cap B) \leq P_p(A) P_p(B).$$

3) "Square root trick"

Let A_1, \dots, A_k be k increasing events. Let $\varepsilon > 0$.

$$\text{Assume } P_p(A_1 \cup \dots \cup A_k) \geq 1 - \varepsilon.$$

Prove that

$$\max_{1 \leq i \leq k} P_p(A_i) \geq 1 - \varepsilon^{1/k}.$$

Exercise 3

Let $X_n := \frac{1}{|B_n|} \sum_{z \in B_n} \mathbb{1}_{z \rightarrow \infty}$. Prove that

$$\lim_{n \rightarrow \infty} X_n = \theta \quad P_p\text{-a.s.}$$